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UNIVERSITY OF ALBERTA

THE ROLE OF NEGATIVE ENERGY WAVES IN PARALLEL SHEAR FLOW INSTABILITY IN HYPERELASTIC FLUID-FILLED TUBES

by

Carmen B. Ropchan

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A THESIS SUBMITTED TO
THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE
IN
APPLIED MATHEMATICS

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

SPRING, 1992

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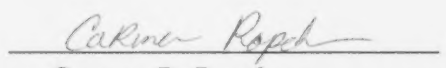

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled THE ROLE OF NEGATIVE ENERGY WAVES IN PARALLEL SHEAR FLOW INSTABILITY IN HYPER-ELASTIC FLUID-FILLED TUBES submitted by CARMEN B. ROPCHAN in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in APPLIED MATHEMATICS.

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Abstract

The stability of fluid flow with step-function velocity and density profiles in a hyperelastic, membraneous tube is examined in both a linear and weakly-nonlinear context. The method of multiple scales is used to derive the three-wave interaction equations for the slowly-varying amplitudes of the plane-wave solutions to the linear problem. It is demonstrated that even in flow configurations of normal-mode stability, the weakly-nonlinear analysis shows that resonant triads of waves can interact in a manner that leads to infinite amplitudes in finite time. The concept of wave energy, defined to be the phase-averaged difference between the energy of the combined mean plus perturbation flow and the energy of the mean flow, is introduced, and it is shown that some linear and weakly-nonlinear instabilities can be explained in terms of the coexistence of modes having wave energies of different signs. Illustrative examples of the various phenomena that occur are demonstrated graphically.

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Table of Contents

Section	Page
Chapter 1: Introduction	1
Chapter 2: The Governing Equations	4
Chapter 3: The Linear Problem	11
Special Solutions of the Dispersion Relation	14
Wave Energy	16
Chapter 4: The O(ϵ) Problem: The Three-Wave Interaction Equations	29
Solutions of the Three-Wave Interaction Equations	33
Chapter 5: Conclusion and Discussion	41
Bibliography	43
Appendix I: Calculations Leading to the O(1) Solutions	46
Appendix II: The O(1) Averaged Energy Equation	48
Appendix III: Calculations Leading to the Three-Wave Interaction Equations	50

List of Figures

Figure **Page**

(3-1) Dispersion curves for $D_1(\omega,k)=0$: $U=2,\ r_0=.8,\ \rho_1=1.02,$
$\rho_2=1.015,\ \gamma=5$ 21

(3-2) Dispersion curves for $D_1(\omega,k)=0$: $U=2,\ r_0=.8,\ \rho_1=1.02,$
$\rho_2=1.015, \gamma=15$ 21

(3-3a) Dispersion curves for $D(\omega,k)=0$: $U=0,\ r_0=.5,\ \rho_1=1.02,$
$\rho_2=1.015,\ \gamma=17,\ \gamma_1=1,\ \gamma_2=1$ 23

(3-3b) Dispersion curves for $D_1(\omega,k)=0$ and $D_2(\omega,k)=0$: $U=0,\ r_0=.5,$
$\rho_1=1.02,\ \rho_2=1.015,\ \gamma=17,\ \gamma_1=1,\ \gamma_2=1$ 23

(3-4a) Dispersion curves for $D(\omega,k)=0$: $U=1,\ r_0=.5,\ \rho_1=1.02,$
$\rho_2=1.015,\ \gamma=17,\ \gamma_1=1,\ \gamma_2=1$ 24

(3-4b) Dispersion curves for $D_1(\omega,k)=0$ and $D_2(\omega,k)=0$: $U=1,\ r_0=.5,$
$\rho_1=1.02,\ \rho_2=1.015,\ \gamma=17,\ \gamma_1=1,\ \gamma_2=1$ 24

(3-5a) Dispersion curves for $D(\omega,k)=0$: $U=2.5,\ r_0=.5,\ \rho_1=1.02,$
$\rho_2=1.015,\ \gamma=17,\ \gamma_1=1,\ \gamma_2=1$ 25

(3-5b) Dispersion curves for $D_1(\omega,k)=0$ and $D_2(\omega,k)=0$: $U=2.5,\ r_0=.5,$
$\rho_1=1.02,\ \rho_2=1.015,\ \gamma=17,\ \gamma_1=1,\ \gamma_2=1$ 25

(3-6a) Dispersion curves for $D(\omega,k)=0$: $U=4,\ r_0=.5,\ \rho_1=1.02,$
$\rho_2=1.015,\ \gamma=17,\ \gamma_1=1,\ \gamma_2=1$ 26

(3-6b) Dispersion curves for $D_1(\omega,k)=0$ and $D_2(\omega,k)=0$: $U=4,\ r_0=.5,$
$\rho_1=1.02,\ \rho_2=1.015,\ \gamma=17,\ \gamma_1=1,\ \gamma_2=1$ 26

(3-7a) Dispersion curves for $D(\omega,k)=0$: $U=6,\ r_0=.5,\ \rho_1=1.02,$
$\rho_2=1.015,\ \gamma=17,\ \gamma_1=1,\ \gamma_2=1$ 27

(3-7b) Dispersion curves for $D_1(\omega,k)=0$ and $D_2(\omega,k)=0$: $U=6,\ r_0=.5,$
$\rho_1=1.02,\ \rho_2=1.015,\ \gamma=17,\ \gamma_1=1,\ \gamma_2=1$ 27

(4-1) Resonant Triads: $U=6,\ r_0=.5,\ \rho_1=1.02,\ \rho_2=1.015,\ \gamma=50,$
$\gamma_1=5,\ \gamma_2=0$ 39

(4-2) Dispersion curves $D(\omega,k)=0$ for resonant triads: $U=6,\ r_0=.5,$
$\rho_1=1.02,\ \rho_2=1.015,\ \gamma=50,\ \gamma_1=5,\ \gamma_2=0$ 40

Chapter 1

Introduction

One of the goals of hydrodynamic stability theory is to predict and explain the conditions under which a steady flow breaks down. Because the equations governing fluid flow, the Navier-Stokes equations, are nonlinear, most of the early work was done in the context of linear stability theory, in which the equations of motion are linearized about some basic, usually steady, state. The resulting equations are then amenable to the powerful tools of Fourier and Laplace transforms. More recently, advances have been made in weakly-nonlinear stability theory. In this type of analysis, the inherent nonlinear behavior of the hydrodynamic fields is taken into account by the addition of small correction terms to the linear solutions. Obtaining these solutions often forms a daunting prospect, since the algebraic manipulations can be formidable. However, once obtained, their properties can often be studied numerically or graphically via computer.

In the general case, a linear stability analysis proceeds as follows. Having obtained simple, though exact, solutions of the fully nonlinear equations, together with any boundary and initial conditions, one superposes small disturbances of the form

$$\hat{A}\exp(i\vec{k}\cdot\vec{x}-i\omega t)+(c.c.),$$

upon the known solutions, where $(c.c.)$ means complex conjugate, and $\hat{A}$ is a function of any spatial variables necessary to satisfy the imposed boundary conditions. Substituting these solutions into the original equations, and neglecting terms of quadratic order or higher in the perturbation fields and their derivatives, one ob-

tains a linear system in the unknown perturbation fields. These can be solved using Fourier and Laplace transforms with respect to $\vec{x}$ and t, respectively. The boundary and/or initial conditions will then determine the allowable frequencies ω as a function of the wavenumber vector $\vec{k}$, where, for a temporal stability analysis, the latter are assumed real, while the former may be complex. Should the solutions $\omega(\vec{k})$ be found to be real for all values of $\vec{k}$, the system is said to be linearly stable with respect to normal-mode disturbances. If complex solutions are found such that $\mathrm{Im}(\omega) > 0$, where $\mathrm{Im}(\cdot)$ denotes the imaginary part of the argument, then the system is said to be linearly unstable. The major drawback of this type of analysis is that normal-mode instability implies exponentially-growing solutions, which is inconsistent with the initial assumption that these terms are small, indeed infinitesimal. A linear stability analysis thus provides information about initial growth rates only. It also forbids wave-wave interactions, for these are inconsistent with the principle of superposition. To study such effects, one must take nonlinearity into account.

In a weakly-nonlinear analysis, the solutions may take the same form as above, but the amplitudes are assumed small-but-finite rather than infinitesimal, so that linear solutions are considered good approximations to the exact solutions. The method employed here depends upon the equations one is dealing with, specifically with the type of nonlinearity present. In the present context, the amplitudes are assumed to be slowly-varying functions of time, with their evolution determined by terms which are quadratic in the linear solutions. An essential feature of the weakly-nonlinear solutions is that, unlike their linear counterparts, they may become infinite in finite time. When this is the case, one must abandon the assumption of weak nonlinearity for sufficiently large times t, and must consider higher-order approximations.

In this presentation, we examine the parallel shear flow of an incompressible,

inviscid fluid in a thin-walled elastic tube. The initial configuration is one similar to that studied by Kelvin and Helmholtz [11,12]; the fluid as a whole is comprised of two immiscible fluids initially in a state of constant relative motion parallel to the axis of the tube. We wish to study the stability of this system when subjected to small plane-wave disturbances that propogate parallel to the axis of the undeformed tube. The presentation proceeds basically as follows.

In Chapter 2, the notation and governing equations are introduced. Asymptotic solutions are proposed as power series in a small positive parameter ϵ, which characterizes the nondimensional amplitude of the perturbation, and the method of multiple scales is used to introduce a 'slow-time' scale to parameterize the evolution of the perturbation amplitudes. In Chapter 3, the linear problem is solved, and the dispersion relation, giving the allowable frequencies as a function of wavenumber, is obtained. While a general stability criterion is made difficult by algebraic complexities, several special cases of solutions of the dispersion relation are examined. Linear instability of two types is shown to occur, and is explained in terms of wave energy. Chapter 4 deals with the solution of the weakly-nonlinear problem. Here, it is shown that triads of waves can interact in an unstable manner, the occurrence of this interaction being dependent upon the energies of the waves in the triad. It is of significance that this nonlinear effect can be predicted on the basis of information that is obtained from the solution of the linear problem. Finally, Chapter 5 contains a brief summary of the results of the previous chapters, and suggests some areas of further study.

Chapter 2

The Governing Equations

The fluid flow is assumed to be taking place in a homogeneous, membraneous, elastic tube that is long relative to the other spatial dimensions under consideration. Cylindrical coordinates are suggested by this configuration, and are so chosen. This selection has its price, however; in Cartesian coordinates, Squire's Theorem [11] guarantees that a normal-mode analysis carried out in two spatial dimensions is sufficient to determine when linear instability occurs in three dimensions. But no analogue of this result has yet been found for two-dimensional flows in cylindrical coordinates, and while it is possible to carry out the linearization process for the general non-axisymmetric problem, the algebra is cumbersome. We shall therefore assume that all fields are axisymmetric.

Longitudinal motion of the tube wall is prevented by the application of tethering forces necessary to prohibit such motion. The properties and behavior of axially-tethered, membraneous tubes form an area of study on their own [7,8], and will not be discussed here. We shall, however, make use of some results of nonlinear elasticity theory in assuming that the tube material is hyperelastic, and thus the motion of the tube wall may be expressed in terms of a strain-energy functional $W(\lambda_1, \lambda_2)$, where λ_1 and λ_2 are the principal stretches in the azimuthal and longitudinal directions, respectively. Lastly, the inertia of the tube wall is assumed negligible compared to that of the fluid; Cowley [7] has shown that this approximation is valid provided that

$$\frac{\rho_m H}{\rho a_0} \ll 1,$$

where ρ, ρ_m, a_0, and H are the fluid and wall densities, and undeformed wall radius and thickness, respectively.

The fluid itself is assumed to be inviscid, incompressible, and discontinuously stratified, having constant densities within and without an interface specified by an initially constant radius. The assumption of axisymmetry does not allow for the consideration of gravitational effects (except parallel to the axis of the undeformed tube), and so the effect of the stratification enters solely via a restoring force at the interface. Under these assumptions, the Navier-Stokes equations reduce to the Euler equations in each region of constant density, with appropriate conditions applied at the interface separating these regions.

The Euler equations for the above problem are

$$(ru)_x + (rv)_r = 0, \tag{2.1}$$

$$u_t + uu_x + vu_r + \rho^{-1}p_x = 0, \tag{2.2}$$

$$v_t + uv_x + vv_r + \rho^{-1}p_r = 0. \tag{2.3}$$

In (2.1)–(2.3), x, r, t, u, v, and p are the dimensional longitudinal and radial coordinates, time, longitudinal and radial velocities, and fluid pressure, respectively, and ρ is the density of the fluid. The appearance of x or t as a subscript denotes partial differentiation with respect to that variable. The conditions on the elastic boundary formed by the tube wall are [25]

$$p = \pi(x,t), \quad r = a_0 + a(x,t), \tag{2.4}$$

$$v = a_t + ua_x, \quad r = a_0 + a(x,t), \tag{2.5}$$

where $a(x,t)$ is the radial deviation of the wall from its undistorted position at $r = a_0$, and $\pi(x,t)$ is the pressure drop due to the wall elasticity. For the cylindrical tube satisfying the conditions stated earlier, π may be written [18]

$$\pi(x,t) = \frac{H}{(a_0+a)(1+e)}\frac{\partial W}{\partial \lambda_1} - \frac{H}{(a_0+a)}\frac{\partial}{\partial x}\left(\frac{a_0 a_x}{(1+a_x^2)^{1/2}}\frac{\partial W}{\partial \lambda_2}\right). \tag{2.6}$$

In the above, the dimensional strain-energy functional is $W(\lambda_1, \lambda_2)$, e is the uniform axial pre-strain, and the azimuthal and longitudinal stretches are given by

$$\lambda_1 = (a_0 + a(x,t))/a_0, \tag{2.7a}$$

$$\lambda_2 = (1+e)(1+a_x^2)^{1/2}. \tag{2.7b}$$

The above system is solved in each homogeneous region separately, and the solutions are matched across the interface so that two conditions are satisfied. The first of these is that the interface is defined by those particles whose radial position is given at all times by $r = r_0 + \eta(x,t)$, where r_0 is the initial position of the interface, and $\eta(x,t)$ its deviation from this position. This is often stated as the requirement that fluid particles on the interface must remain there, and is expressed mathematically as

$$\frac{D}{Dt}(r - r_0 - \eta(x,t)) = 0, \quad \text{on } r = r_0 + \eta(x,t),$$

where, as usual,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$$

is the substantial-differentiation operator. Carrying out the differentiation, one obtains

$$v = \eta_t + u\eta_x, \quad \text{on } r = r_0 + \eta(x,t). \tag{2.8}$$

This is, in fact, the same condition that gives rise to (2.5), and forbids the formation of 'holes' within the fluid.

The second condition is one that governs the change in pressure across the interface. When the fluid surface between two immiscible fluids is deformed, there is a discontinuity in the normal stress at the interface arising from the restoring force that acts to minimize its surface area. The force per unit area is given by

Laplace's formula [15] as

$$p_{res} = \gamma(R_1^{-1} + R_2^{-1}),$$

where R_1 and R_2 are the principal radii of curvature of the interface, and γ is the surface tension parameter. If $\vec{n}$ is a unit normal pointing away from the axis of the cylinder, then differential geometry [14] gives the result that $R_1^{-1} + R_2^{-1} = \nabla \cdot \vec{n}$. The above pressure condition may then be written

$$p_o - p_i = -\gamma \nabla \cdot \vec{n}, \tag{2.9}$$

where p_o and p_i denote the fluid pressures just outside and inside the fluid interface, respectively. Since $r = r_0 + \eta(x,t)$ specifies the fluid interface, it follows that

$$\vec{n} = (-\eta_x, 0, 1)/(1 + \eta_x^2)^{1/2}. \tag{2.10}$$

It is customary to introduce non-dimensional coordinates and fields to render solutions independent of the physical dimensions of a particular problem. To this end, introduce the non-dimensional (asterisked) quantities as follows:

$$\left.\begin{aligned} (x,r) &= a_0(x^*, r^*), \\ (u,v) &= c_0(u^*, v^*), \\ t &= (a_0/c_0)t^*, \\ p &= \rho_0 c_0^2 p^*, \\ W &= W_0 W^*, \end{aligned}\right\} \tag{2.11}$$

where

$$\begin{aligned} \rho &= \rho_0 \rho^*, \\ c_0^2 &= \frac{H W_0}{a_0 \rho_0}, \\ \gamma &= (\rho_0 a_0 c_0^2)\gamma^*, \end{aligned}$$

and ρ_0 is a reference density. From this point forward, we will assume that the non-dimensionalization specified by (2.11) has been carried out, and will omit asterisks in the equations that result.

It is trivial to verify that the following is an exact solution of the fully nonlinear system consisting of the non-dimensional analogues of (2.1)-(2.5), (2.8), and (2.9):

$$\left.\begin{array}{rcl} (u_0, v_0, \rho, p_0) & = & \begin{cases} (U,\ 0,\ \rho_1,\ P + \gamma/r_0), & r < r_0, \\ (0,\ 0,\ \rho_2,\ P), & r > r_0, \end{cases} \\ \eta_0 & = & a_0 \ = \ 0 \end{array}\right\} \tag{2.12}$$

with

$$P \ = \ \frac{1}{(1+e)} \left(\frac{\partial W(1, 1+e)}{\partial \lambda_1} \right),$$

and U, ρ_1, and ρ_2 are constants. Clearly, $\vec{u_0}$ satisfies

$$\nabla \times \vec{u_0} = \vec{0}, \quad \vec{u_0} = (u_0, 0, v_0);$$

such flows are said to be irrotational. Kelvin, in 1869, proved that if the initial motion of an incompressible, inviscid fluid is irrotational, then so is any subsequent motion. Hence, if $\vec{u}$ is the velocity field at any later time, then $\nabla \times \vec{u} = \vec{0}$, and thus there exists a velocity potential $\phi(r, x, t)$ such that $\vec{u} = \nabla \phi$. The continuity equation (2.1) then implies that $\nabla^2 \phi = 0$. Further, (2.2) and (2.3) may be integrated to give Bernoulli's equation in each homogeneous region

$$p \ = \ -\rho \left(\phi_t + \frac{1}{2} (\nabla \phi)^2 + C(t) \right). \tag{2.13}$$

These equations will be used instead of (2.1)–(2.3), with the velocity field obtained via $\vec{u} = \nabla \phi$.

We now suppose that the state given by (2.12) is perturbed slightly by primed quantities, so that

$$\left.\begin{array}{rcl} \phi & = & u_0 x + \epsilon \phi', \ \ \vec{u} \ = \ \nabla \phi, \\ (a, \eta) & = & \epsilon (a', \eta'), \\ p & = & p_0 + \epsilon p'. \end{array}\right\} \tag{2.14}$$

In (2.14), $0 < \epsilon \ll 1$ is a non-dimensional parameter characteristic of the amplitudes of the disturbances, so that primed quantities and their derivatives are assumed to be $O(1)$. Substitution of these solutions in the nonlinear equations, and the boundary and interfacial conditions given earlier, shows that nonlinearity becomes significant over a time-scale of $O(\epsilon^{-1})$. This suggests using the method of multiple scales [2] to introduce a 'slow-time' scale[1] $T = \epsilon t$; temporal derivatives will transform as

$$\partial_t \longmapsto \partial_t + \epsilon \partial_T,$$

with t and T henceforth treated as independent variables.

Following Swaters [25], the perturbation fields will be assumed to be given asymptotically by power series in ϵ:

$$\begin{aligned} f' &\sim \sum_{n=0}^{\infty} \epsilon^n f^{(n)}(r, x, t; T), \\ g' &\sim \sum_{n=0}^{\infty} \epsilon^n g^{(n)}(x, t; T), \end{aligned}$$

with $f' \in \{\phi', p'\}$ and $g' \in \{a', \eta'\}$. We thus obtain the following system, where primes have been omitted:

$$\nabla^2(\phi^{(0)} + \epsilon\phi^{(1)}) + O(\epsilon^2) = 0, \tag{2.15a}$$

$$\begin{aligned} p^{(0)} + \epsilon p^{(1)} &= -\rho\{\phi_t^{(0)} + u_0\phi_x^{(0)} \\ &\quad + \epsilon\{\phi_t^{(1)} + u_0\phi_x^{(1)} + \phi_T^{(0)} + \frac{1}{2}(\nabla\phi^{(0)})^2\}\} + O(\epsilon^2), \end{aligned} \tag{2.15b}$$

with the expanded (about $\epsilon = 0$) boundary and interfacial conditions:

$$\begin{aligned} \phi_r^{(0)} + \epsilon\phi_r^{(1)} &= \eta_t^{(0)} + u_0\eta_x^{(0)} \\ &\quad + \epsilon\{\eta_t^{(1)} + u_0\eta_x^{(1)} + \eta_T^{(0)} + \phi_x^{(0)}\eta_x^{(0)} \end{aligned} \tag{2.16a}$$

[1] In principle, one could also introduce a 'long-space' scale as well, in order to describe long spatial modulations.

$$- \eta^{(0)}\phi^{(0)}_{rr}\} + O(\epsilon^2), \quad \text{on } r = r_0,$$

$$\left[p^{(0)}\right] + \epsilon\left[p^{(1)}\right] = \gamma\{\eta^{(0)}_{xx} + (\eta^{(0)}/r_0^2) \tag{2.16b}$$
$$+ \epsilon\{\eta^{(1)}_{xx} + (\eta^{(1)}/r_0^2) + (\eta^{(0)^2}_x/2r_0) - (\eta^{(0)^2}/r_0^3)\}\}$$
$$- \epsilon\eta^{(0)}\left[p^{(0)}_r\right] + O(\epsilon^2), \quad \text{on } r = r_0,$$

$$\phi^{(0)}_r + \epsilon\phi^{(1)}_r = a^{(0)}_t + \epsilon\{a^{(1)}_t + a^{(0)}_T + \phi^{(0)}_x a^{(0)}_x - a^{(0)}\phi^{(0)}_{rr}\} \tag{2.17a}$$
$$+ O(\epsilon^2), \quad \text{on } r = 1,$$

$$p^{(0)} + \epsilon p^{(1)} = (W^0_{11} - W^0_1)a^{(0)}/(1+e) - W^0_2 a^{(0)}_{xx} \tag{2.17b}$$
$$+ \epsilon\{(W^0_{11} - W^0_1)a^{(1)}/(1+e) - W^0_2 a^{(1)}_{xx}$$
$$+ (W^0_1 - W^0_{11} + (W^0_{111}/2))a^{(0)^2}/(1+e) - (W^0_{12}/2)a^{(0)^2}_x$$
$$+ (W^0_2 - W^0_{12})a^{(0)}a^{(0)}_{xx} - a^{(0)}p^{(0)}_r\} + O(\epsilon^2), \quad \text{on } r = 1.$$

In the above,

$$[f] = f(r_0^+, x, t; T) - f(r_0^-, x, t; T), \tag{2.18}$$

and

$$W^0_{i_1 i_2 \cdots i_n} = \left(\frac{\partial^n W(1, 1+e)}{\partial\lambda_{i_1}\partial\lambda_{i_2}\cdots\partial\lambda_{i_n}}\right). \tag{2.19}$$

We now turn our attention to the solution of the $O(1)$ problem.

Chapter 3

The Linear Problem

For convenience, we reproduce the $O(1)$ problem here:

$$\nabla^2\phi^{(0)} = 0, \tag{3.1a}$$

$$p^{(0)} = -\rho(\phi_t^{(0)} + u_0\phi_x^{(0)}), \tag{3.1b}$$

$$\phi_r^{(0)} = \eta_t^{(0)} + u_0\eta_x^{(0)}, \quad \text{on } r = r_0, \tag{3.2a}$$

$$\left[p^{(0)}\right] = \gamma(\eta_{xx}^{(0)} + (\eta^{(0)}/r_0^2)), \quad \text{on } r = r_0, \tag{3.2b}$$

$$\phi_r^{(0)} = a_t^{(0)}, \quad \text{on } r = 1, \tag{3.3a}$$

$$p^{(0)} = \gamma_1 a^{(0)} - \gamma_2 a_{xx}^{(0)}, \quad \text{on } r = 1. \tag{3.3b}$$

The elastic parameters γ_1 and γ_2 are defined by

$$\gamma_1 = (W_{11}^0 - W_1^0)/(1+e), \tag{3.4a}$$

$$\gamma_2 = W_2^0, \tag{3.4b}$$

and are known to be positive [18]. Solving (3.1a) by the method of separation of variables, we assume a travelling-wave solution of the form

$$\phi^{(0)}(r,x,t;T) = \mathrm{Re}\left\{\phi(r;T)\exp(ikx - i\omega t)\right\}, \tag{3.5}$$

where the wavenumber k is assumed real, the frequency ω is allowed to be complex, $i^2 = -1$, and $\mathrm{Re}\{\cdot\}$ denotes the real part of the argument. Substitution of (3.5) into (3.1a) shows that the radial dependence of the velocity potential is governed by the modified Bessel equation of order zero in the variable $|k|r$

$$\phi_{rr} + r^{-1}\phi_r - k^2\phi = 0.$$

The general solution may thus be written

$$\varphi^{(0)}(r,x,t;T) = \begin{cases} \mathrm{Re}\{A(T)I_0(|k|r)\exp(i\theta)\}, & r < r_0, \\ \mathrm{Re}\{(B(T)I_0(|k|r) + C(T)K_0(|k|r))\exp(i\theta)\}, & r > r_0, \end{cases} \quad (3.6)$$

where $\theta = kx - \omega t$ is the characteristic, or phase, variable. We have used the fact that $K_0(x)$ has a logarithmic singularity at $x = 0$ to omit this part of the general solution in the interior of the fluid. Henceforth, it will be understood that only the real part of a complex-valued function corresponds to a physical field, and we drop the Re notation from such expressions.

It is clear from the remaining equations that the (x,t) dependence of $\eta^{(0)}$ and $a^{(0)}$ will be the same as that of $\phi^{(0)}$, and we may therefore write

$$\eta^{(0)} = A_1(T)\exp(i\theta), \quad (3.7a)$$

$$a^{(0)} = A_2(T)\exp(i\theta). \quad (3.7b)$$

Conditions (3.2) and (3.3) may then be used to express A, B, and C as functions of A_1 and A_2 (see Appendix I for details):

$$A = \frac{-i(\omega - Uk)A_1}{|k|I_1(|k|r_0)}, \quad (3.8a)$$

$$B = \frac{i\omega(A_2K_1(|k|r_0) - A_1K_1(|k|))}{|k|P(k)}, \quad (3.8b)$$

$$C = \frac{i\omega(A_2I_1(|k|r_0) - A_1I_1(|k|))}{|k|P(k)}, \quad (3.8c)$$

with

$$P(k) = K_1(|k|)I_1(|k|r_0) - K_1(|k|r_0)I_1(|k|). \quad (3.9)$$

To evaluate the pressure conditions, we note that, from (3.1b)

$$p^{(0)} = \begin{cases} i\rho_1(\omega - Uk)AI_0(|k|r)\exp(i\theta), & r < r_0, \\ i\rho_2\omega(BI_0(|k|r) + CK_0(|k|r))\exp(i\theta) & r > r_0, \end{cases}$$

Hence, the pressure jump (3.2b) becomes

$$i\rho_2\omega(BI_0(|k|r_0) + CK_0(|k|r_0)) - i\rho_1(\omega - Uk)AI_0(|k|r_0) = \gamma(r_0^{-2} - k^2)A_1.$$

With the aid of (3.8), this may be written

$$D_1(\omega,k)A_1 + \Lambda(\omega,k)A_2 \;=\; 0, \tag{3.10}$$

where

$$\begin{aligned} D_1(\omega,k) &= r_0\{\rho_1(\omega - Uk)^2 I_0(|k|r_0)/(|k|I_1(|k|r_0)) \\ &\quad - \rho_2\omega^2 F(k)/(|k|P(k)) + \gamma(r_0^{-2} - k^2)\}, \end{aligned} \tag{3.11}$$

$$\Lambda(\omega,k) = \rho_2\omega^2/k^2P(k), \tag{3.12}$$

$$F(k) = K_1(|k|)I_0(|k|r_0) + I_1(|k|)K_0(|k|r_0). \tag{3.13}$$

Waves centred on the undeformed fluid interface have frequencies ω which satisfy $D_1(\omega,k) = 0$ when the elastic boundary is replaced by a rigid one; it is thus the dispersion relation for that problem. Similarly, (3.3b) gives

$$i\rho_2\omega(BI_0(|k|) + CK_0(|k|)) \;=\; (\gamma_1 + k^2\gamma_2)A_2;$$

by virtue of (3.8), this may be written

$$D_2(\omega,k)A_2 + \Lambda(\omega,k)A_1 \;=\; 0, \tag{3.14}$$

with

$$D_2(\omega,k) = -\rho_2\omega^2 G(k)/(|k|P(k)) - (\gamma_1 + k^2\gamma_2), \tag{3.15}$$

$$G(k) = K_1(|k|r_0)I_0(|k|) + I_1(|k|r_0)K_0(|k|). \tag{3.16}$$

Modes centred on $r = 1$ satisfy $D_2(\omega,k) = 0$ when the interface at $r = r_0$ is replaced by a rigid surface.

Eliminating A_2 (or A_1) between (3.10) and (3.14), the dispersion relation for the problem at hand is obtained as

$$D(\omega,k) \;=\; D_1(\omega,k)D_2(\omega,k) - \Lambda^2(\omega,k) \;=\; 0. \tag{3.17}$$

The dispersion relation is thus a quartic in ω; the solutions give the frequencies as possibly complex-valued functions of k and the physical parameters. When the ω are all real, the unperturbed system is said to be linearly stable with respect to normal-mode disturbances. Otherwise, solutions will occur in complex-conjugate pairs, and so one solution will give rise to an exponentially-growing wave. The system is thus classified as linearly unstable. At this stage, one would generally calculate these roots, and attempt to find conditions under which they are real or complex; this can be done for (3.17) using either solutions by radicals [1] or in terms of Weierstrass Elliptic Functions [13]. However, the coefficients of ω are lengthy expressions, and the significance of the various parameters in determining when instability occurs is lost in the algebraic expressions for the roots of $D(\omega, k) = 0$. We shall therefore attempt to understand the physical mechanisms at work by considering a few special cases.

Special Solutions of the Dispersion Relation

$U = 0$: In the absence of shear, (3.17) may be written

$$(ac - e)\omega^4 + (ad + bc)\omega^2 + bd = 0, \tag{3.18}$$

where

$$\begin{aligned} a &= r_0\{\rho_1 I_0(|k|r_0)/[|k|I_1(|k|r_0)] - \rho_2 F(k)/[|k|P(k)]\}, \\ b &= \gamma(r_0^{-2} - k^2), \\ c &= -\rho_2 G(k)/[|k|P(k)], \\ d &= -(\gamma_1 + k^2\gamma_2), \\ e &= \rho_2^2/[k^4 P^2(k)]. \end{aligned}$$

This is quadratic in ω^2, and the roots are easily found to be

$$\frac{-(ad + bc) \pm ((ad - bc)^2 + 4bde)^{1/2}}{2(ac - e)}. \tag{3.19}$$

It follows that linear instability is predicted whenever the discriminant of (3.19) is negative, or the roots themselves are negative. The discriminant, regarded as a quadratic in b, may be shown to be always positive, and hence we need only concern ourselves with conditions for which (3.18) are real, but negative. Because $I_0(x)$, $I_1(x)$ and $K_0(x)$, $K_1(x)$ are, respectively, monotonically increasing and decreasing functions of x, it is easily shown that $e - ac < 0$. Thus, linear instability occurs when the numerator of (3.19) is negative; this is so for $|k| < r_0^{-1}$, provided $\gamma \neq 0$ (if $\gamma = 0$, the roots of (3.18) are clearly non-negative). Surface tension thus acts to destabilize the flow for all modes having wavelengths $\lambda = 2\pi/|k|$ which exceed the circumference of the undistorted interface. This is the same stability criterion obtained by Rayleigh [20] in his study of the instability of a liquid jet in air; it occurs because surface tension acts to decrease the surface area of the jet (or interface, in the context of the present problem), thereby releasing surface energy.

Pipe Flow: When the elastic boundary is replaced by a rigid one, conditions (3.3) are replaced by the condition that the radial component of velocity must vanish on $r = 1$. As mentioned previously, (3.17) then reduces to $D_1(\omega, k) = 0$, where $D_1(\omega, k)$ is given by (3.11). The roots are easily computed, and instability is predicted whenever the discriminant is negative, which occurs if either $|k| \leq r_0^{-1}$ or $|k| > r_0^{-1}$ and

$$\gamma \;<\; \frac{\rho_1\rho_2|k|F(k)I_0(|k|r_0)U^2}{(\rho_1 P(k)I_0(|k|r_0) - \rho_2 F(k)I_1(|k|r_0))(r_0^{-2} - k^2)}.$$

Thus, the surface tension acts to suppress the instability for sufficiently short waves.

Short-Wave Approximation: For those wavenumbers satisfying $|k|r_0 \gg 1$, one may write the terms appearing in (3.17) asymptotically as

$$\begin{aligned} D_1(\omega,k)D_2(\omega,k) &\sim r_0(\rho_1(\omega - Uk)^2 + \rho_2\omega^2 - \gamma|k|^3)(\rho_2\omega^2 - \gamma_2|k|^3)/k^2, \\ \Lambda^2(\omega,k) &\sim \frac{4r_0\rho_2^2\omega^4}{k^2}\exp(-2|k|(1-r_0)). \end{aligned}$$

The interaction term is thus exponentially subdominant to the other terms, and so $D(\omega,k) \sim D_1(\omega,k)D_2(\omega,k)$. Asymptotically, then, the frequencies are readily verified to be real, provided $\gamma \neq 0$, and complex if $\gamma = 0$.

The assumption of a homogeneous fluid, whether miscible ($\gamma = 0$) or immiscible ($\gamma \neq 0$) does not result in sufficient simplification of (3.17) to easily compute the roots. Similarly, an asymptotic treatment in the case $|k|r_0 \ll 1$ shows that all terms in (3.17) contribute equally, again resulting in no real simplification.

It is clear that the linear problem is rich and complex; an exhaustive parameter study would be a lengthy procedure. It is fortunate, however, that for the range of parameters considered here, the essential characteristics of the solutions to (3.17) appear to be well-described by the solutions of $D_1(\omega,k)D_2(\omega,k) = 0$; the interaction term does not seem to have a great effect. We shall see this as we examine the role of wave energy in predicting instability.

Wave Energy

The use of wave energy to study both linear and nonlinear instabilities in hydrodynamic systems originates from its use in the study of microwave tubes and plasmas. It is relatively well-known that if $\varepsilon(\omega,k) = 0$ is the linear dispersion relation for plane electrostatic waves in a uniform medium, then the wave energy for neutrally-stable (ω real) waves is [27]

$$E \;=\; \omega\frac{\partial\varepsilon}{\partial\omega}\cdot(\text{electric field energy}).$$

This expression is evaluated at the chosen (ω,k) pair satisfying $\varepsilon(\omega,k) = 0$. The wave energy in this sense is the difference between the energy of the system when the wave is present, and its energy when the wave is absent. A wave is thus defined to have negative energy if its establishment lowers the total energy of the plasma.

The nature of some instabilities, both linear and nonlinear, is critically dependent upon the relative signs of the wave energies. This is significant in that one can predict some nonlinear instabilities from a linear analysis.

In 1979, Cairns [4] proved that similar ideas could be exploited in the study of some plane parallel shear flows. In his consideration of a three-layer fluid with step-function velocity and density profiles, it was shown that the wave energy, averaged over one wavelength, could be written

$$E = \frac{1}{4}\omega\frac{\partial D}{\partial \omega}\cdot(\text{wave amplitude})^2, \tag{3.20}$$

where $D(\omega, k) = 0$ is the linear dispersion relation. Further, the Kelvin - Helmholtz instability that occurs in this problem was explained in terms of positive and negative energy waves coexisting on a single interface, and another type of linear instability was shown qualitatively to exist when a positive energy mode propagated on one interface, and a negative energy mode on the other. Craik and Adam [9], in a subsequent paper, verified the correctness of Cairns' prediction; they also carried out a weakly-nonlinear analysis of the same problem, and showed that a type of instability known in plasma physics as 'explosive' instability could take place among triads of waves.

It is our goal here to show that (3.20) also holds in the present context. To verify this, we will confirm, by direct calculation, that the average wave energy for neutrally-stable waves on the fluid interface is given by

$$E = \frac{1}{4}\omega\frac{\partial D}{\partial \omega}|\epsilon A_1|^2,$$

with E evaluated at $D(\omega, k) = 0$, written in the form

$$D(\omega, k) = D_1(\omega, k) - \frac{\rho_2^2\omega^4}{k^4 P^2(k) D_2(\omega, k)}.$$

Upon differentiating the above with respect to ω, and using the fact that

$D(\omega, k) = 0$, one obtains

$$E = \frac{1}{4}\left\{-4D_1 + \frac{D_1}{D_2}\omega\frac{\partial D_2}{\partial \omega} + \omega\frac{\partial D_1}{\partial \omega}\right\}|\epsilon A_1|^2. \tag{3.21}$$

For the fluid interior to the interface at $r = r_0 + \epsilon\eta$, the contribution to the kinetic energy due to the presence of the $O(1)$ wave is

$$\begin{aligned} KE_{int} &= \frac{\rho_1}{2}\int_0^{r_0+\epsilon\eta^{(0)}} r((U+\epsilon u^{(0)})^2 + \epsilon^2 v^{(0)^2} - U^2)dr \\ &= \frac{\rho_1\epsilon}{2}\int_0^{r_0} r(2Uu^{(0)} + \epsilon(u^{(0)^2} + v^{(0)^2}))dr \\ &\quad + \frac{\rho_1 r_0\eta^{(0)}\epsilon^2}{2}(2Uu^{(0)} + \epsilon(u^{(0)^2} + v^{(0)^2}))_{r=r_0} + O(\epsilon^3), \end{aligned}$$

where Taylor's Theorem, followed by the Leibnitz formula for differentiating an integral, has been used. Upon substitution of the $O(1)$ solutions from (3.6), and recalling that only the real parts of these expressions are used, the above becomes

$$\begin{aligned} KE_{int} &= \frac{\rho_1\epsilon^2}{4}\int_0^{r_0} r\left(k^2|A|^2(I_0^2(|k|r) + I_1^2(|k|r))\right)dr \\ &\quad + \frac{i\rho_1 U r_0 k I_0(|k|r_0)\epsilon^2}{4}(A_1^* A - A^* A_1) \\ &\quad + \{\text{terms in } \exp(\pm in\theta),\ n = 1,2,3\} + O(\epsilon^3). \end{aligned}$$

Integrating over one wavelength ($x \in [0, 2\pi/|k|]$), terms in the fast-phase oscillations average to zero. Substitution for A in terms of A_1 from (3.8), and carrying out the integration over r, one obtains, to $O(\epsilon^2)$,

$$\begin{aligned} KE_{int} &= \frac{\rho_1 r_0}{4}\left\{\frac{2kU(\omega - Uk)I_0(|k|r_0) + (\omega - Uk)^2 I_0(|k|r_0)}{|k|I_1(|k|r_0)}\right\}|\epsilon A_1|^2 \\ &= \frac{1}{4}\left\{\omega\frac{\partial D_1}{\partial \omega} - D_1 + \frac{\rho_2 r_0\omega^2 F(k)}{|k|P(k)} + r_0\gamma(r_0^{-2} - k^2)\right\}|\epsilon A_1|^2. \end{aligned} \tag{3.22}$$

By performing a similar calculation for the fluid exterior to the fluid interface, one finds the contribution to the kinetic energy in this region to be, again to $O(\epsilon^2)$,

$$KE_{ext} = -\frac{1}{4}\left\{\frac{\rho_2 r_0\omega^2 F(k)}{|k|P(k)} + \frac{\rho_2\omega^2 G(k)D_1}{|k|P(k)D_2} + 2D_1\right\}|\epsilon A_1|^2. \tag{3.23}$$

Finally, the contribution of the potential energy is (see Appendix II)

$$PE = \frac{1}{4}\left((\gamma_1 + k^2\gamma_2)\frac{D_1}{D_2} - r_0\gamma(r_0^{-2} - k^2)\right)|\epsilon A_1|^2 + O(\epsilon^3). \qquad (3.24)$$

The total energy is the sum of (3.22)-(3.24), and is verified to be equal to the expression given by (3.21).

To demonstrate the applications of the energy written in this form, we first consider the behavior of waves on each interface separately. As we shall demonstrate graphically, normal-mode instability will occur for $U \neq 0$ only when modes have energies of opposite sign for a given value of k.

Modes centred on $r = 1$ have frequencies ω which satisfy $D_2(\omega, k) = 0$, and the energy of any such wave is readily verified to be

$$E = \frac{1}{4}\omega\frac{\partial D_2}{\partial \omega}|\epsilon A_2|^2 = -\frac{\rho_2\omega^2 G(k)}{2|k|P(k)}|\epsilon A_2|^2 > 0.$$

It would thus be predicted that these modes do not interact in a destabilizing manner. This is easily seen to be true; the roots of $D_2(\omega, k) = 0$ are always real.

However, let us suppose that ω_1 and ω_2 are real roots of $D_1(\omega, k) = 0$, for some value of k. Writing $D_1(\omega, k) = C_0(\omega - \omega_1)(\omega - \omega_2)$, where C_0 is a constant, the energy of these modes is

$$E_1 = \frac{1}{4}\omega_1\left(\frac{\partial D_1}{\partial \omega}\right)_{\omega_1}|\epsilon A_1|^2 = \frac{1}{4}C_0\omega_1(\omega_1 - \omega_2)|\epsilon A_1|^2,$$

$$E_2 = \frac{1}{4}\omega_2\left(\frac{\partial D_1}{\partial \omega}\right)_{\omega_2}|\epsilon A_1|^2 = \frac{1}{4}C_0\omega_2(\omega_2 - \omega_1)|\epsilon A_1|^2.$$

Whence, if ω_1 and ω_2 are of the same sign, then their energies are of opposite sign. Now, for $\gamma \neq 0$, the discriminant of the solutions of $D_1(\omega, k) = 0$ is found to be a monotonically increasing function of $|k|$, passing through zero, for $\gamma \neq 0$, at some number $|k| = k_c$ for given values of the physical parameters. For large $|k|r_0$, the solutions behave asymptotically as

$$\frac{\rho_1 kU \pm |k|^{3/2}\sqrt{\gamma(\rho_1 + \rho_2)}}{(\rho_1 + \rho_2)}.$$

Provided $U \neq 0$, it follows that there exists k_0 such that $\mathrm{sgn}(\omega_1) = \mathrm{sgn}(\omega_2)$ for all $k \in (k_c, k_0)$, and thus, on a graph of $\mathrm{Re}(\omega)$ vs k, instability would be predicted near $|k| = k_c$. This instability is demonstrated in Figures (3-1) and (3-2) for the selected values of the physical parameters, and $\gamma = 5$ and $\gamma = 15$, respectively. It is found that the energy for modes on the lower branch is negative respectively for $k \in [1.76, 2.04]$ and $k \in [1.44, 1.52]$, approximately, and positive elsewhere when the solutions are real. Wave energies for modes on the upper branches are always positive in both cases. Note that an increase in γ has the effect of stabilizing some modes; however, those modes that remain complex experience an increase in growth rate, given by $|\mathrm{Im}(\omega)|$, with an increase in γ.

Consider now the complete dispersion relation

$$D(\omega, k) \;=\; D_1(\omega, k) D_2(\omega, k) - \rho_2^2 \omega^4 / k^4 P^2(k) \;=\; 0,$$

where we will assume that the modes are only weakly coupled; i.e. $|k| r_0$ is not too small. Let ω_1 and ω_2 be neutrally-stable solutions of $D_1(\omega, k) = 0$ and $D_2(\omega, k) = 0$, respectively, and suppose that these solutions are close:

$$\omega_2 \;=\; \omega_1 + \delta, \qquad |\delta / \omega_1| \ll 1.$$

A solution ω close to the above roots satisfying $D(\omega, k) = 0$ may be written

$$\omega \;=\; \omega_1 + \Delta, \qquad |\Delta / \omega_1| \ll 1,$$

where, since Δ may be complex, $|\cdot|$ denotes modulus in the above. Then

$$D_1(\omega, k) D_2(\omega, k) - \rho_2^2 \omega^4 / k^4 P^2(k) \;=\; 0,$$

which may be written approximately as

$$\Delta \left(\frac{\partial D_1}{\partial \omega} \right)_{\omega_1} (\Delta - \delta) \left(\frac{\partial D_2}{\partial \omega} \right)_{\omega_2} - \frac{\rho_2^2 \omega_1^4}{k^4 P^2(k)} \;=\; 0,$$

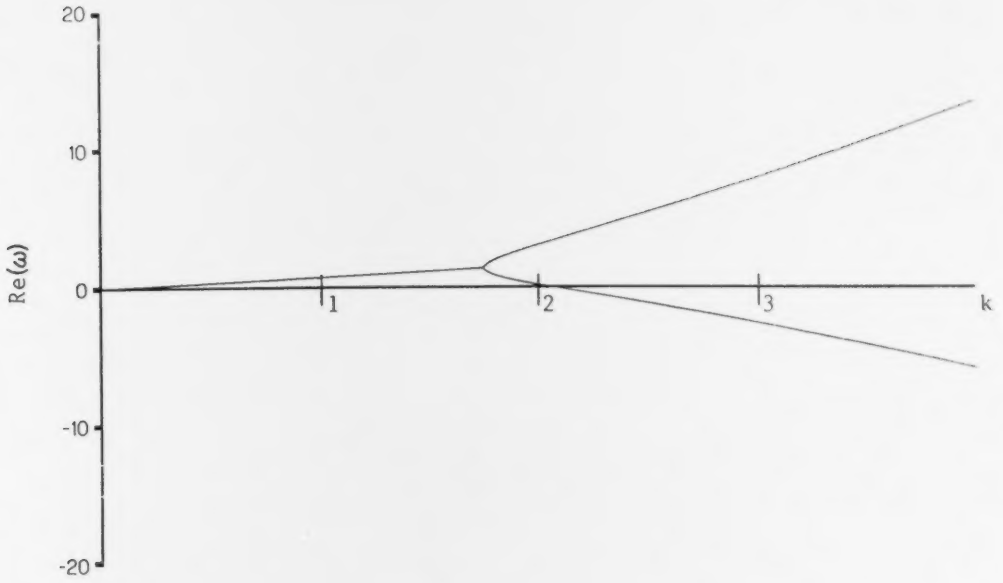

Figure (3-1): Dispersion curves for $D_1(\omega, k) = 0$: $U = 2,\ r_0 = .8,\ \rho_1 = 1.02,$ $\rho_2 = 1.015,\ \gamma = 5$

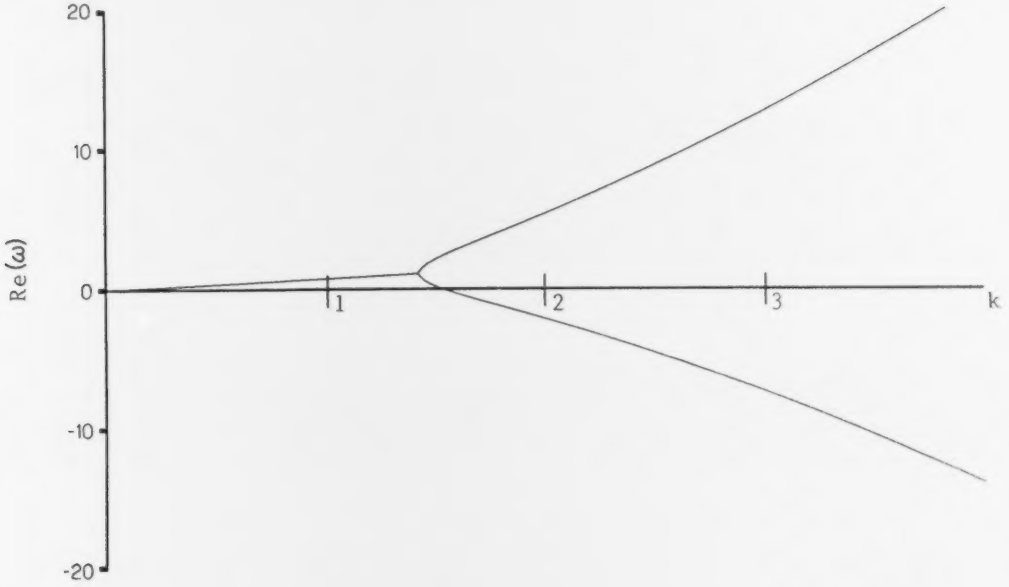

Figure (3-2): Dispersion curves for $D_1(\omega, k) = 0$: $U = 2,\ r_0 = .8,\ \rho_1 = 1.02,$ $\rho_2 = 1.015,\ \gamma = 15$

upon using Taylor's Theorem. This is a quadratic in Δ, and may be written in the form

$$\Delta^2 - \delta\Delta - T = 0,$$

$$T = \frac{\rho_2^2\omega_1^4}{k^4P^2(k)}\left(\frac{\partial D_1}{\partial\omega_1}\right)^{-1}\left(\frac{\partial D_2}{\partial\omega_2}\right)^{-1} \approx \frac{\rho_2^2\omega_1^6}{k^4P^2(k)}\left(\omega_1\frac{\partial D_1}{\partial\omega_1}\right)^{-1}\left(\omega_2\frac{\partial D_2}{\partial\omega_2}\right)^{-1}.$$

The roots Δ are easily seen to be real if T is positive, which occurs when the wave energies associated with ω_1 and ω_2 are of the same sign. The roots are complex, implying instability, if $|\delta| < 2|T|^{1/2}$, when $T < 0$. It is to be noted that the instability so predicted arises from the coexistence of waves on different interfaces, but that it can be predicted on the basis of analyzing the dispersion properties of waves on each interface separately, under the assumption of weak coupling. This is of great computational value, in particular for multi-layered systems, where each interface increases the degree of the dispersion relationship by two.

The onset of this type of instability, known as 'reactive' instability in the parlance of plasma physics, is demonstrated graphically in Figures (3-3)–(3-7) for various values of U. The dispersion curves for $D_1(\omega,k) = D_2(\omega,k) = 0$ are also displayed in order to assess the effects of the interaction term. Note that the ordinate axis begins at $k = 1$; in both cases, the curves increase monotonically from zero in the almost-linear manner that occurs for $k \in [1,2]$. The interaction term had very little effect on the roots of $D_1(\omega,k) = D_2(\omega,k) = 0$, except near points of intersection of these curves.

In Figure (3-3a), we see the linear instability for $|k| \leq r_0^{-1}$ that is predicted by (3.19). The roots of $D_1(\omega,k) = D_2(\omega,k) = 0$ are relatively well-spaced except near the point where this instability ceases. However, all modes have positive energy near these points, and the only effect of the interaction is to cause an exchange of identities of the roots near the intersection point. For sufficiently

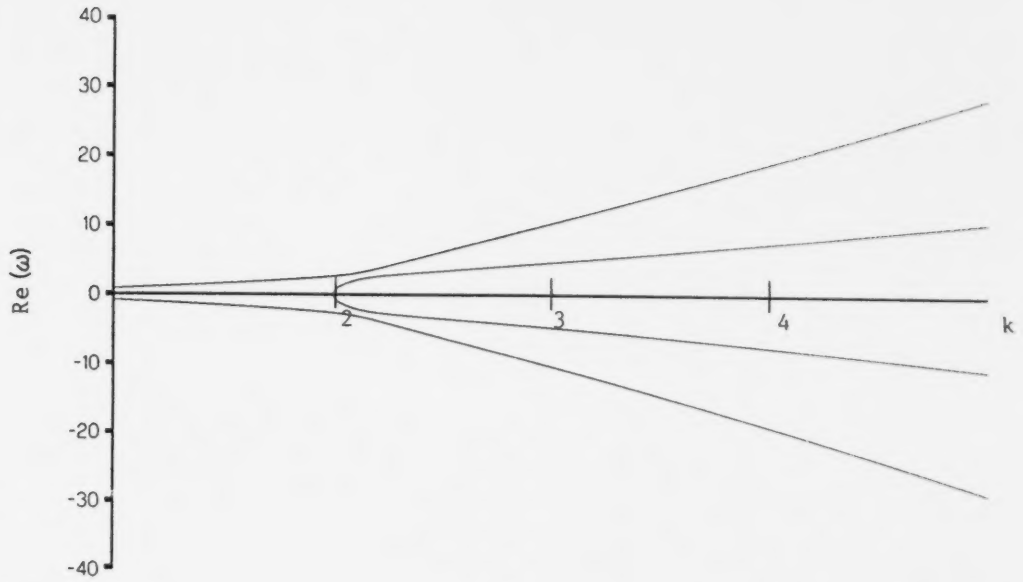

Figure (3-3a): Dispersion curves for $D(\omega, k) = 0$: $U = 0,\ r_0 = .5,\ \rho_1 = 1.02,$ $\rho_2 = 1.015,\ \gamma = 17,\ \gamma_1 = 1,\ \gamma_2 = 1$

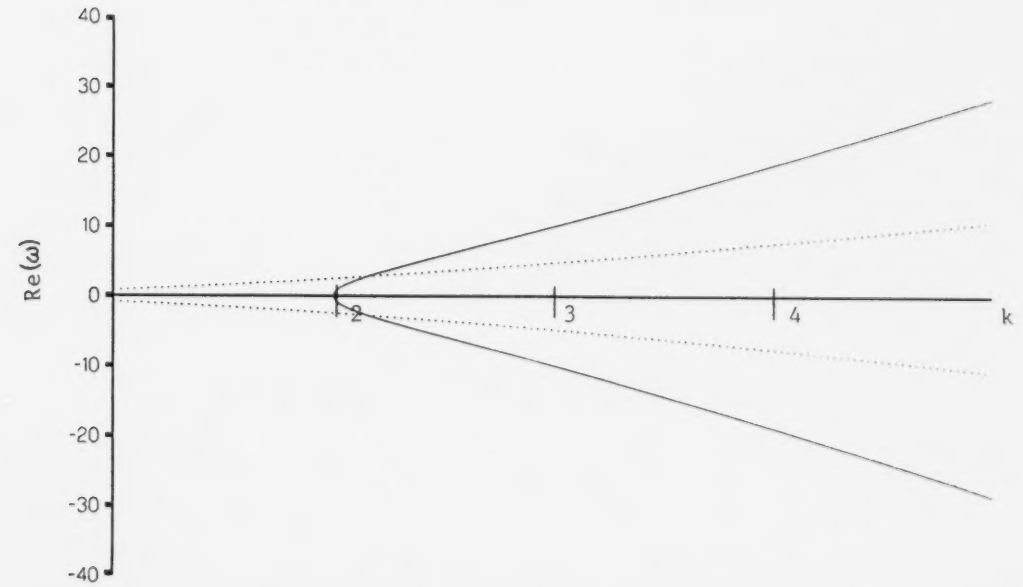

Figure (3-3b): Dispersion curves for $D_1(\omega, k) = 0$ (————) and $D_2(\omega, k) = 0$ (·····): $U = 0,\ r_0 = .5,\ \rho_1 = 1.02,\ \rho_2 = 1.015,$ $\gamma = 17,\ \gamma_1 = 1,\ \gamma_2 = 1$

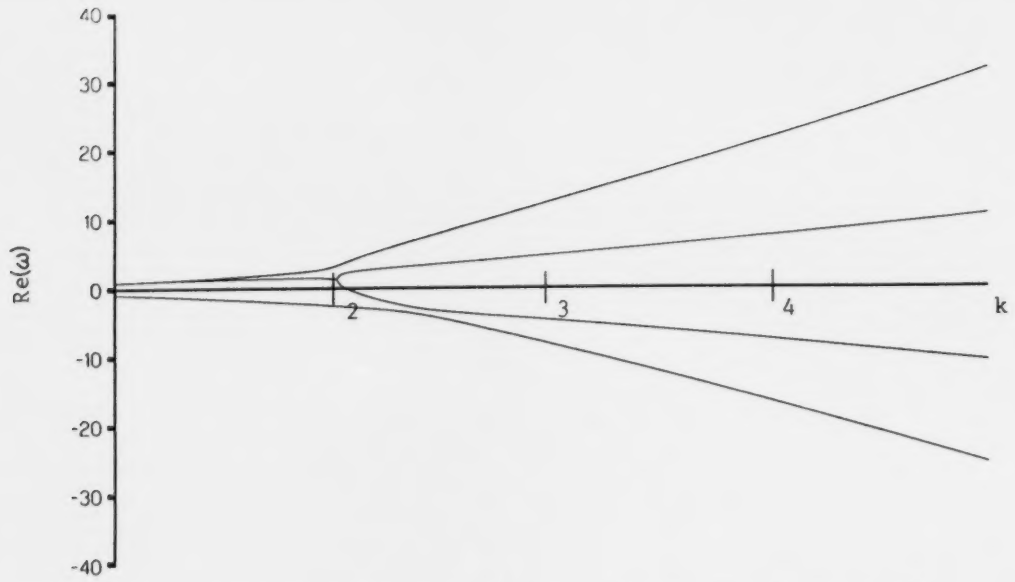

Figure (3-4a): Dispersion curves for $D(\omega, k) = 0$: $U = 1,\ r_0 = .5,\ \rho_1 = 1.02,$ $\rho_2 = 1.015,\ \gamma = 17,\ \gamma_1 = 1,\ \gamma_2 = 1$

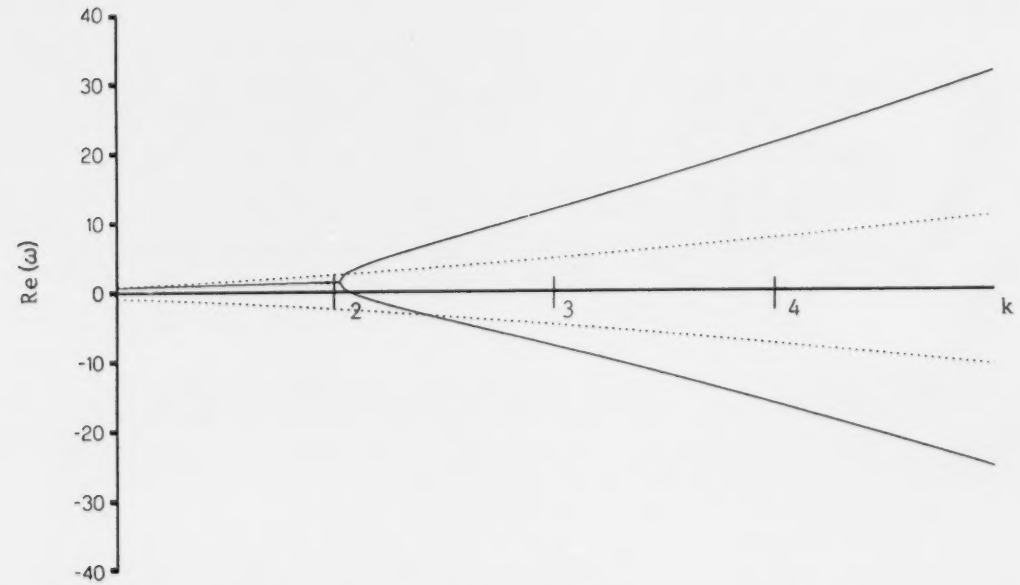

Figure (3-4b): Dispersion curves for $D_1(\omega, k) = 0$ (————) and $D_2(\omega, k) = 0$ (·····): $U = 1,\ r_0 = .5,\ \rho_1 = 1.02,\ \rho_2 = 1.015,$ $\gamma = 17,\ \gamma_1 = 1,\ \gamma_2 = 1$

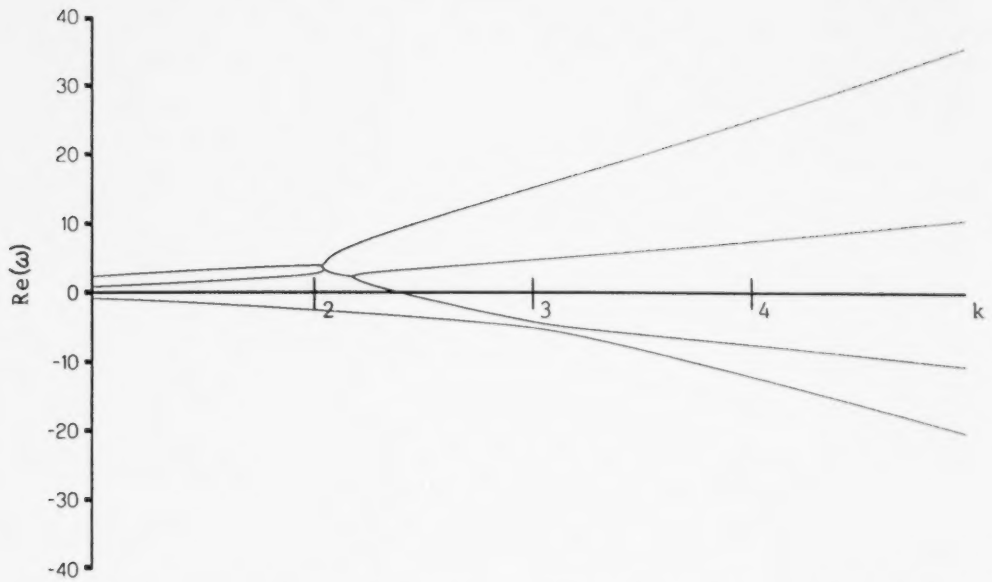

Figure (3-5a): Dispersion curves for $D(\omega, k) = 0$: $U = 2.5,\ r_0 = .5,\ \rho_1 = 1.02,$ $\rho_2 = 1.015,\ \gamma = 17,\ \gamma_1 = 1,\ \gamma_2 = 1$

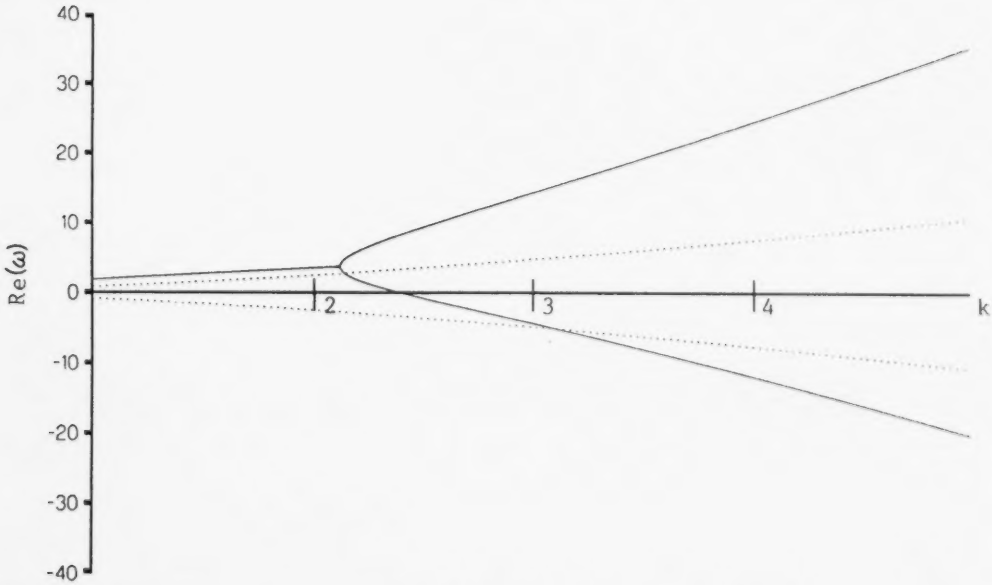

Figure (3-5b): Dispersion curves for $D_1(\omega, k) = 0$ (————) and $D_2(\omega, k) = 0$ (·····): $U = 2.5,\ r_0 = .5,\ \rho_1 = 1.02,\ \rho_2 = 1.015,$ $\gamma = 17,\ \gamma_1 = 1,\ \gamma_2 = 1$

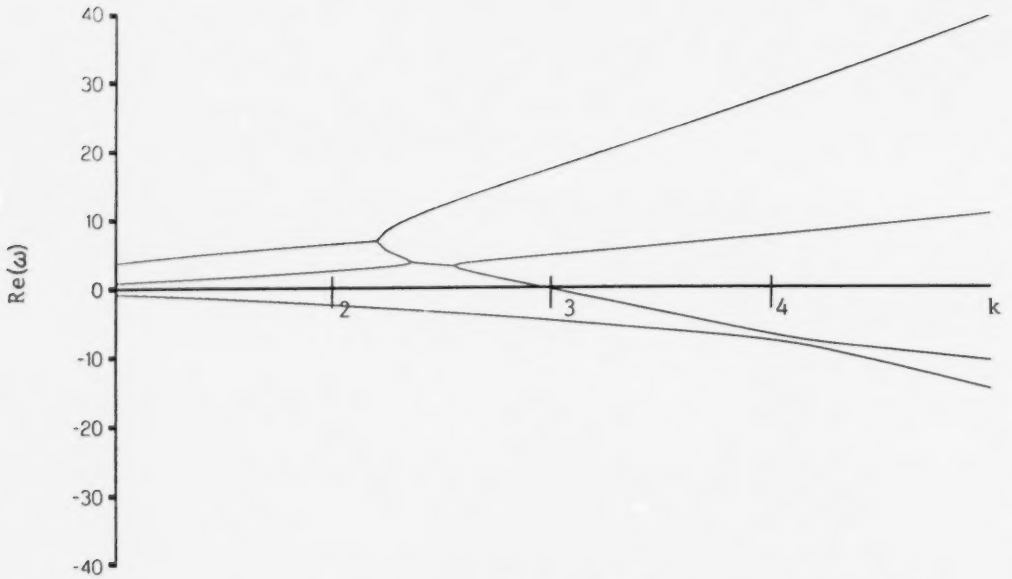

Figure (3-6a): Dispersion curves for $D(\omega, k) = 0$: $U = 4,\ r_0 = .5,\ \rho_1 = 1.02,$ $\rho_2 = 1.015,\ \gamma = 17,\ \gamma_1 = 1,\ \gamma_2 = 1$

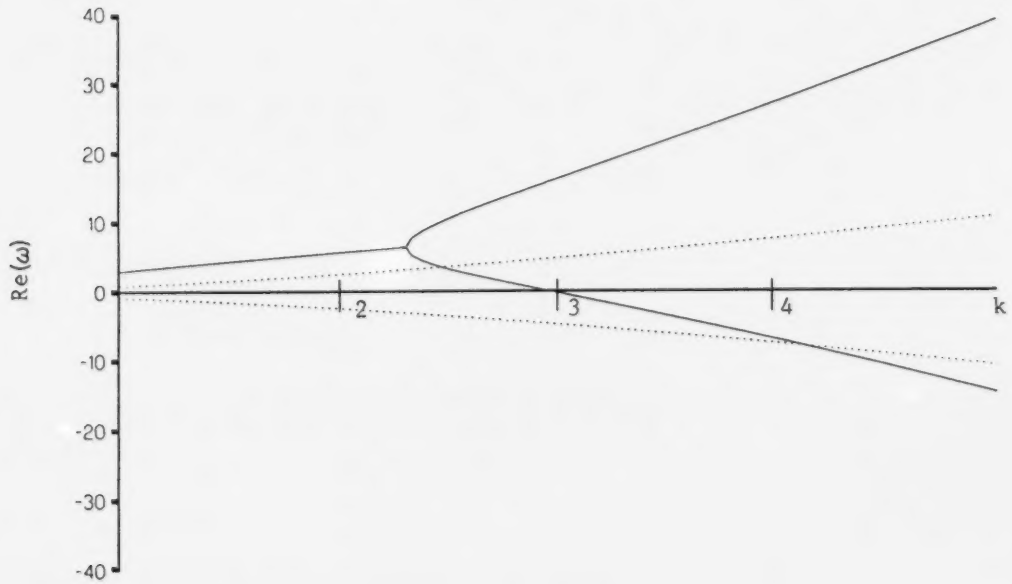

Figure (3-6b): Dispersion curves for $D_1(\omega, k) = 0$ (————) and $D_2(\omega, k) = 0$ (·····): $U = 4,\ r_0 = .5,\ \rho_1 = 1.02,\ \rho_2 = 1.015,$ $\gamma = 17,\ \gamma_1 = 1,\ \gamma_2 = 1$

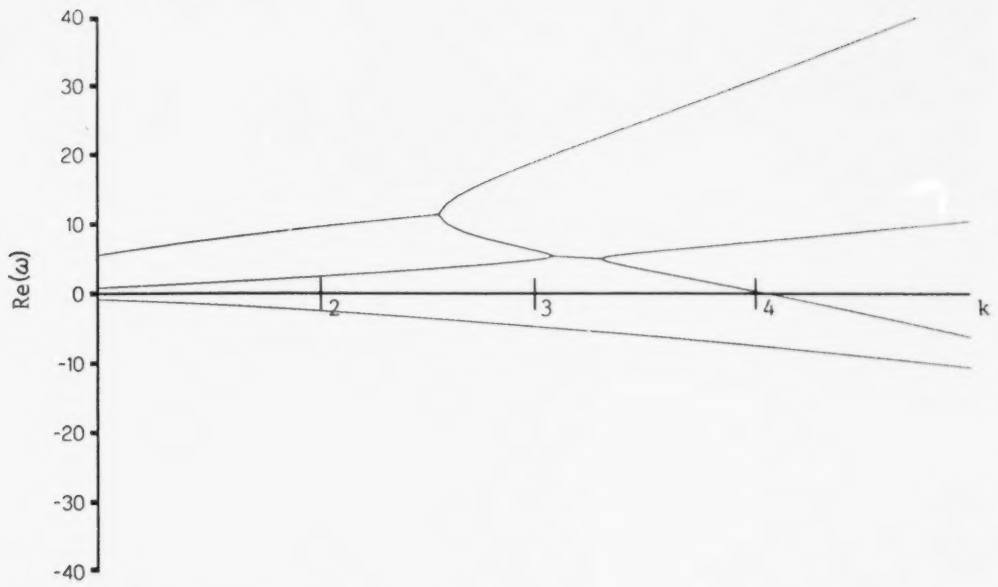

Figure (3-7a): Dispersion curves for $D(\omega, k) = 0$: $U = 6,\ r_0 = .5,\ \rho_1 = 1.02,$ $\rho_2 = 1.015,\ \gamma = 17,\ \gamma_1 = 1,\ \gamma_2 = 1$

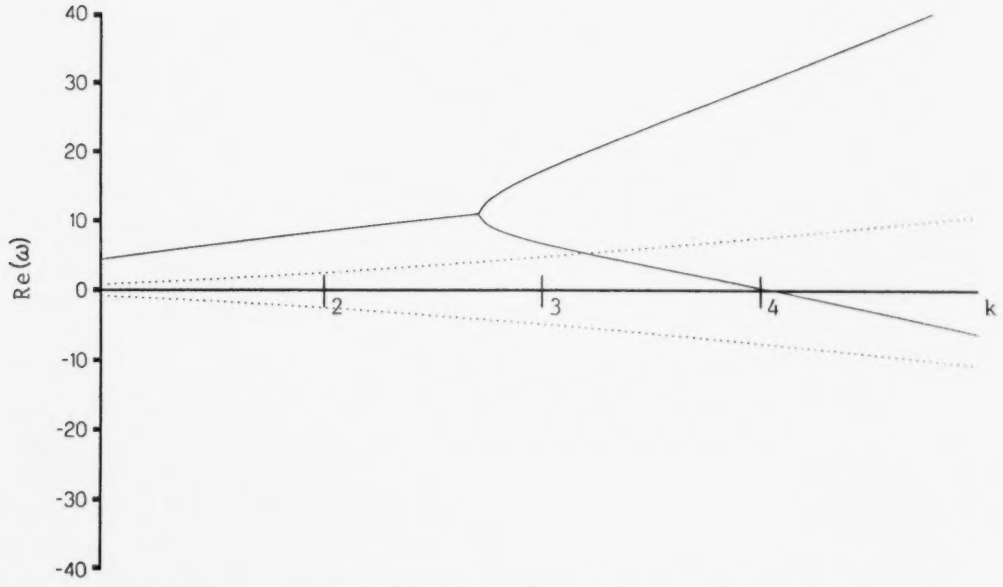

Figure (3-7b): Dispersion curves for $D_1(\omega, k) = 0$ (————) and $D_2(\omega, k) = 0$ (·····): $U = 6,\ r_0 = .5,\ \rho_1 = 1.02,\ \rho_2 = 1.015,$ $\gamma = 17,\ \gamma_1 = 1,\ \gamma_2 = 1$

large k, the minimum and maximum frequencies satisfy $D_1(\omega, k) \approx 0$, while the solutions between these satisfy $D_2(\omega, k) \approx 0$. In Figure (3-4a), the shear has been increased to unity; there is very little change in the dispersion curves, except for a slight upward displacement of the unstable modes. All neutrally-stable solutions have positive energy, except for the lower branch of the unstable modes. These have negative energy in the interval $k \in [2.025, 2.065]$, and positive energy for $k \geq 2.07$, approximately. Thus, the interaction near the intersection point at $k \approx 2.4$ is of the passive sort seen previously. It is interesting to note that the interaction term $\Lambda^2(\omega, k)$ has a slight stabilizing effect of those modes near $k = r_0^{-1}$. This becomes more pronounced as the shear is increased.

For $U = 2.5$, Figure (3-5a) shows that the real part of the unstable solutions has continued its upward trend, lying now above the other solutions. It is seen that near the first point of intersection at $k \approx 2.05$, the interaction between the modes is of the reactive type; the energy of the lower unstable branch is negative between these two regions of instability. It is also negative for $k \in [2.195, 2.395]$, but this condition is not sustained for sufficiently large k to cause reactive instability near the other intersection point. Figures (3-6) and (3-7) show that as the shear is increased, the region of reactive instability becomes more pronounced, and displays a shift towards larger values of k.

The behavior of the solutions for other values of the physical parameters was also investigated, and it was found that when reactive instability occurred, it appeared in a manner similar to that displayed here.

We have thus seen that energy considerations may be used to predict the onset of linear instability. In the next section, we will investigate how these same concepts can be used to study the weakly-nonlinear interactions of the $O(1)$ waves, and predict instability that can occur as a result of these interactions.

Chapter 4
The $O(\epsilon)$ Problem:
The Three-Wave Interaction Equations

Isolating the weakly-nonlinear, or $O(\epsilon)$, terms in (2.15)–(2.17), the following system is obtained:

$$\nabla^2\phi^{(1)} = 0, \tag{4.1a}$$

$$p^{(1)} = -\rho(\phi_t^{(1)} + u_0\phi_x^{(1)} + \phi_T^{(0)} + \frac{1}{2}(\phi_x^{(0)^2} + \phi_r^{(0)^2})), \tag{4.1b}$$

$$\phi_r^{(1)} = \eta_t^{(1)} + u_0\eta_x^{(1)} + \eta_T^{(0)} + \phi_x^{(0)}\eta_x^{(0)} - \eta^{(0)}\phi_{rr}^{(0)}, \quad \text{on } r = r_0, \tag{4.2a}$$

$$\left[p^{(1)}\right] = \gamma(\eta_{xx}^{(1)} + (\eta^{(1)}/r_0^2) + (\eta_x^{(0)^2}/2r_0) - (\eta^{(0)^2}/r_0^3)) - \eta^{(0)}\left[p_r^{(0)}\right], \quad \text{on } r = r_0, \tag{4.2b}$$

$$\phi_r^{(1)} = a_t^{(1)} + a_T^{(0)} + \phi_x^{(0)}a_x^{(0)} - a_0\phi_{rr}^{(0)}, \quad \text{on } r = 1, \tag{4.3a}$$

$$p^{(1)} = \gamma_1 a^{(1)} - \gamma_2 a_{xx}^{(1)} + \gamma_3 a^{(0)^2} + \gamma_4 a^{(0)}a_{xx}^{(0)} + \gamma_5 a_x^{(0)^2} - a^{(0)}p_r^{(0)}, \quad \text{on } r = 1, \tag{4.3b}$$

where

$$\gamma_3 = (W_1^0 - W_{11}^0 + (W_{111}^0/2))/(1+e),$$

$$\gamma_4 = W_2^0 - W_{12}^0,$$

$$\gamma_5 = -W_{12}^0/2,$$

and $[p^{(1)}]$ and $[p_r^{(0)}]$ are the pressure jumps as given by (2.18). To study the interaction of resonant triads of waves, the $O(1)$ fields will be written as the superposition of three neutrally-stable waves of the form given by (3.7a). Thus,

$$\eta^{(0)} = \sum_{n=1}^{3} A_1^{(n)} \exp(ik_n x - i\omega_n t), \tag{4.4}$$

with analogous expressions for the other linear fields. The relationships between the various coefficients, as given by (3.8), are still valid for each n, and each pair (ω_n, k_n) is a solution of $D(\omega, k) = 0$. Such a triad is said to be in resonance if any of the following conditions hold:

$$\pm\theta_1 \pm \theta_2 \pm \theta_3 \;=\; 0,$$

where $\theta_n = k_n x - \omega_n t$ is the phase of the n'th wave. It will be assumed that the waves in (4.4) satisfy such a condition. Owing to the symmetry of the above in (4.1)–(4.3), we may assume, without loss of generality, that $\theta_1 = \theta_2 + \theta_3$; i.e., that

$$k_1 \;=\; k_2 + k_3, \quad \omega_1 \;=\; \omega_2 + \omega_3. \tag{4.5}$$

As we will see, the nature of the interactions between these waves is critically dependent upon the relative signs of the wave energies.

Upon examining the linear terms in (4.1)–(4.3), the $O(\epsilon)$ solutions will be of the same form as the solutions to the linear problem. These will be written

$$\phi^{(1)}(r, x, t; T) \;=\; \begin{cases} \sum_{n=1}^{3} \hat{A}^{(n)} I_0(|k_n| r) \exp(i\theta_n), & r < r_0, \\ \sum_{n=1}^{3} (\hat{B}^{(n)} I_0(|k_n| r) + \hat{C}^{(n)} K_0(|k_n| r)) \exp(i\theta_n), & r > r_0, \end{cases} \tag{4.6}$$

$$\eta^{(1)}(x, t; T) \;=\; \sum_{n=1}^{3} \hat{A}_1^{(n)} \exp(i\theta_n), \tag{4.7a}$$

$$a^{(1)}(x, t; T) \;=\; \sum_{n=1}^{3} \hat{A}_2^{(n)} \exp(i\theta_n). \tag{4.7b}$$

The analysis proceeds exactly as in the linear problem. When the above are substituted into the weakly-nonlinear system, (4.2)–(4.3) will yield five equations which are linear in the amplitudes appearing in (4.6) and (4.7), and allow them to be expressed as functions which are linear in $A_{1_T}^{(n)}$ and quadratic in the $O(1)$ amplitudes. As in Chapter 3, the unknown amplitudes $\hat{A}^{(n)}$, $\hat{B}^{(n)}$, and $\hat{C}^{(n)}$ will be expressed as functions of the interfacial and boundary amplitudes $\hat{A}_1^{(n)}$ and $\hat{A}_2^{(n)}$. Written in this form, we will obtain the equations that govern the slow-time evolution of the $A_1^{(n)}$.

The details of the calculations may be found in Appendix III, although we reproduce the main results here:

$$\hat{A}^{(n)} = \frac{-i(\omega_n - Uk_n)\hat{A}_1^{(n)} + A_{1_T}^{(n)} + R(\theta_n)}{|k_n| I_1(|k_n| r_0)}, \tag{4.8a}$$

$$\hat{B}^{(n)} = \frac{K_1(|k_n|)\sigma_n - K_1(|k_n| r_0)\tau_n}{|k_n| P(k_n)}, \tag{4.8b}$$

$$\hat{C}^{(n)} = \frac{I_1(|k_n|)\sigma_n - I_1(|k_n| r_0)\tau_n}{|k_n| P(k_n)}, \tag{4.8c}$$

where

$$\sigma_n = -i\omega_n \hat{A}_1^{(n)} + A_{1_T}^{(n)} + S(\theta_n),$$

$$\tau_n = -i\omega_n \hat{A}_2^{(n)} + A_{2_T}^{(n)} + T(\theta_n).$$

These hold for $n = 1, 2, 3$, with $P(k_n)$ given by (3.9), and $R(\theta_n)$, $S(\theta_n)$, and $T(\theta_n)$ are those quantities in the nonlinear terms that have phase θ_n in (4.2a) and (4.3a), respectively, as given by (III.5a), (III.6a), and (III.7a). Referring again to Appendix III, application of the pressure condition (4.2b) yields

$$D_2(n)\hat{A}_2^{(n)} + \Lambda(n)\hat{A}_1^{(n)} = \chi(n)A_{1_T}^{(n)} + \Omega(n), \tag{4.9a}$$

where the notation $f(n) = f(\omega_n, k_n)$ will be used for the remainder of the chapter. The condition governing the change in pressure across the elastic boundary, (4.3b), gives

$$D_1(n)\hat{A}_1^{(n)} + \Lambda(n)\hat{A}_2^{(n)} = \Phi(n)A_{1_T}^{(n)} + \Gamma(n). \tag{4.9b}$$

In the above, D_1, D_2, and Λ have their usual meanings from (3.11), (3.15), and (3.12), respectively. The coefficients $\chi(n)$, $\Omega(n)$, $\Phi(n)$, and $\Gamma(n)$ are given by (III.13) and (III.15).

It will be recalled that the wavenumber/frequency pairs (k_n, ω_n) are neutrally-stable solutions of $D(\omega, k) = D_1(\omega, k)D_2(\omega, k) - \Lambda^2(\omega, k) = 0$. Whence, dividing

(4.9b) by $D_1(n)$ and (4.9a) by $\Lambda(n)$ and subtracting, one finds

$$(\Phi(n)\Lambda(n) - D_1(n)\chi(n))A_{1_T}^{(n)} = D_1(n)\Omega(n) - \Lambda(n)\Gamma(n), \tag{4.10}$$

which is the evolution equation for the n'th $O(1)$ amplitude. In Appendix III, it is shown that

$$\Phi(n)\Lambda(n) - D_1(n)\chi(n) = -i\Lambda(n)\frac{\partial D(n)}{\partial \omega_n}, \tag{4.11}$$

while

$$D_1(n)\Omega(n) - \Lambda(n)\Gamma(n) = \Lambda(n)\lambda\beta(n). \tag{4.12}$$

In (4.12),

$$\begin{aligned}
\beta(1) &= A_1^{(2)}A_1^{(3)}, \\
\beta(2) &= A_1^{(1)}A_1^{(3)^*}, \\
\beta(3) &= A_1^{(1)}A_1^{(2)^*},
\end{aligned}$$

where * denotes complex conjugation. The interaction coefficient λ is independent of n, and is given by (III.21). Substitution of (4.11) and (4.12) into (4.10) yield the three-wave interaction equations:

$$-i\frac{\partial D(1)}{\partial \omega_1}A_{1_T}^{(1)} = \lambda A_1^{(2)}A_1^{(3)}, \tag{4.13a}$$

$$-i\frac{\partial D(2)}{\partial \omega_2}A_{1_T}^{(2)} = \lambda A_1^{(1)}A_1^{(3)^*}, \tag{4.13b}$$

$$-i\frac{\partial D(3)}{\partial \omega_3}A_{1_T}^{(3)} = \lambda A_1^{(1)}A_1^{(2)^*}. \tag{4.13c}$$

Conservation of wave energy follows immediately from these equations; since

$$\begin{aligned}
\left(\omega_1\frac{\partial D(1)}{\partial \omega_1}|A_1^{(1)}|^2\right)_T &= \omega_1\frac{\partial D(1)}{\partial \omega_1}(A_1^{(1)}A_{1_T}^{(1)^*} + A_1^{(1)^*}A_{1_T}^{(1)}) \\
&= -2\omega_1 \mathrm{Re}(i\lambda A_1^{(1)}A_1^{(2)^*}A_1^{(3)^*}),
\end{aligned}$$

with similar expressions holding for $A_1^{(2)}$ and $A_1^{(3)}$, it follows that

$$\left(\sum_{i=1}^{3}\omega_i\frac{\partial D(i)}{\partial\omega_i}|A_1^{(i)}|^2\right)_T = -2(\omega_1-\omega_2-\omega_3)\mathrm{Re}(i\lambda A_1^{(1)}A_1^{(2)^*}A_1^{(3)^*}).$$

By virtue of the resonance conditions (4.5),

$$(E_1+E_2+E_3)_T = 0. \tag{4.14}$$

It is clear from (4.14) that if the signs of the wave energies are the same, the wave amplitudes cannot grow simultaneously. Conversely, should one of the waves have energy of different sign from the other two, it is consistent with energy conservation that the amplitude of this mode could increase/decrease without bound, while the amplitudes of the others grew in the opposite sense. We shall prove that this is indeed the case by solving (4.13).

Solutions of the Three-Wave Interaction Equations

The interaction equations derived above are functionally identical to those that arise in the study of three-wave interactions among electrostatic waves in a rarefied plasma. The general solutions have been derived by Coppi et. al. [6], and we will therefore omit some of the details of the calculations here.

Introduce the change of variables

$$\begin{aligned}
A_1^{(1)}(T) &= \frac{b_1(T)}{|\lambda_2\lambda_3|^{1/2}}\exp(i\psi_1(T)),\\
A_1^{(2)}(T) &= \frac{b_2(T)}{|\lambda_1\lambda_3|^{1/2}}\exp(i\psi_2(T)),\\
A_1^{(3)}(T) &= \frac{b_3(T)}{|\lambda_1\lambda_2|^{1/2}}\exp(i\psi_3(T)),
\end{aligned}$$

where $b_j(T)$ and $\psi_j(T)$ are real-valued, and

$$\lambda_j = -\lambda\left(\frac{\partial D(j)}{\partial\omega_j}\right)^{-1}.$$

Then (4.13) may be written as the real system

$$b_{1_T} = -s_1 b_2 b_3 \sin\psi, \tag{4.15a}$$

$$b_{2_T} = s_2 b_1 b_3 \sin\psi, \tag{4.15b}$$

$$b_{3_T} = s_3 b_1 b_2 \sin\psi, \tag{4.15c}$$

$$\psi_T = \left(\frac{-s_1 b_2 b_3}{b_1} + \frac{s_2 b_1 b_3}{b_2} + \frac{s_3 b_1 b_2}{b_3}\right)\cos\psi, \tag{4.16}$$

with $s_j = \mathrm{sgn}(\lambda_j)$ and $\psi = \psi_1 - \psi_2 - \psi_3$. It follows readily from the first three of these equations that we have the following constants of motion, the so-called Manley-Rowe relations:

$$s_1 b_1^2(T) + s_2 b_2^2(T) = M_1, \tag{4.17a}$$

$$s_1 b_1^2(T) + s_3 b_3^2(T) = M_2. \tag{4.17b}$$

If (4.16) is multiplied by $\sin\psi$ and one uses (4.15) in the result, one also finds the conserved quantity

$$b_1(T) b_2(T) b_3(T) \cos\psi(T) = \Gamma. \tag{4.17c}$$

The constant Γ is characteristic of the strength of the interactions between the resonating waves. By this it is meant that if $\psi(0) = \pi/2$, then (4.17c) guarantees that $\psi(T) = \pi/2$ for arbitrary initial amplitudes, and thus the interactions described by (4.15) are at their maximum. The Manley-Rowe relations are closely related to energy conservation.

We now seek to show that the amplitudes of the waves become infinite in finite time if $b_1^2(0)$ is sufficiently large, and the sign of E_1 is opposite to that of E_2 and E_3. If this is not the case, so that $s_1 = s_2 = s_3$, then (4.17a,b) guarantee that the $b_j(T)$ remain bounded for all T. We therefore take $s_2 = s_3 = 1$, $s_1 = s = \pm 1$, and use (4.15)–(4.17) to write

$$(b_1^2)_T = \pm 2 b_1 b_2 b_3 \sin\psi \tag{4.18}$$

$$= \pm 2(b_1^2 b_2^2 b_3^2 - \Gamma^2)^{1/2}$$
$$= \pm 2\left[(b_1^2 - \beta_1)(b_1^2 - \beta_2)(b_1^2 - \beta_3)\right]^{1/2},$$

where $\beta_3 \geq \beta_2 \geq \beta_1$ when the roots are all real. The above cubic must remain non-negative for all T, and so any unbounded solution $b_1^2(T)$ must satisfy $b_1^2(0) \geq \beta_3$ (there is always at least one non-negative root of (4.18), since $\beta_1\beta_2\beta_3 = \Gamma^2$). Bearing this restriction in mind, we observe that (4.18) is a separable equation, and so we may write

$$T = \pm \frac{1}{2} \int_{b_1^2(0)}^{b_1^2(T)} \frac{d(b^2)}{[(b^2 - \beta_1)(b^2 - \beta_2)(b^2 - \beta_3)]^{1/2}}, \tag{4.19}$$

which is an Elliptic Integral and may be inverted to give $b_1^2(T)$. To evaluate (4.19), we distinguish between the cases when the β_j are all equal, when at least two are distinct and real, and when $\beta_3 \geq 0$ and β_1 and β_2 are conjugate complex.

i) $\beta_1 = \beta_2 = \beta_3 = \beta$

In this symmetric case, (4.19) is easily solved to give

$$b_1^2(T) = \beta + (\mp T + (b_1^2(0) - \beta)^{-1/2})^{-2}.$$

The time of explosion is thus

$$T_{exp} = (b_1^2(0) - \beta)^{-1/2}.$$

ii) β_1, β_2, β_3 all real, at least two distinct

Let $y^2 = (\beta_3 - \beta_1)/(b^2 - \beta_1)$ and $m^2 = (\beta_2 - \beta_1)/(\beta_3 - \beta_1)$. Then (4.19) may be written

$$T = \mp(\beta_3 - \beta_1)^{-1/2} \int_{y(b_1^2(0))}^{y(b_1^2(T))} (1 - y^2)^{-1/2}(1 - m^2 y^2)^{-1/2} dy,$$

which is the difference of two Elliptic Integrals (of the first kind), and may be solved using [1], among others. Solving for $b_1^2(T)$ gives

$$b_1^2(T) = \beta_1 + \frac{(\beta_3 - \beta_1)}{\mathrm{sn}^2[(\beta_3 - \beta_1)^{1/2}(\mp T + T_{exp}); m]},$$

$$T_{exp} = \frac{1}{(\beta_3-\beta_1)^{1/2}}\mathrm{sn}^{-1}\left[\left(\frac{\beta_3-\beta_1}{b_1^2(0)-\beta_1}\right)^{1/2};m\right],$$

where $\mathrm{sn}[x;m]$ denotes the Jacobi Elliptic Function with modulus m. When $\beta_3 > \beta_2 > \beta_1$, the time of explosion is given by T_{exp}. When $\beta_3 = \beta_2 > \beta_1$, then $m = 1$; since $\mathrm{sn}[x;1] = \tanh(x)$, the above may be rewritten as

$$\begin{aligned} b_1^2(T) &= \beta_3 + (\beta_3-\beta_1)\mathrm{cosech}^2((\beta_3-\beta_1)^{1/2}(\mp T + T_{exp})), \\ T_{exp} &= \frac{1}{(\beta_3-\beta_1)^{1/2}}\tanh^{-1}\left(\frac{\beta_3-\beta_1}{b_1^2(0)-\beta_1}\right)^{1/2}. \end{aligned}$$

Finally, when $\beta_3 > \beta_2 = \beta_1$, $\mathrm{sn}[x,0] = \sin(x)$, and we obtain

$$\begin{aligned} b_1^2(T) &= \beta_3 + (\beta_3-\beta_1)\cot^2((\beta_3-\beta_1)^{1/2}(\mp T + T_{exp})), \\ T_{exp} &= \frac{1}{(\beta_3-\beta_1)^{1/2}}\sin^{-1}\left(\frac{\beta_3-\beta_1}{b_1^2(0)-\beta_1}\right)^{1/2}. \end{aligned}$$

iii) $\beta_3 = \beta \geq 0$, $\beta_1 = \beta_2^*$

Write the cubic in (4.19) as

$$(b^2-\beta_1)(b^2-\beta_2)(b^2-\beta_3) = (b^2-\beta)(b^4-2ab^2+c),$$

where $c > a^2$, and introduce

$$\begin{aligned} y &= \frac{G+\beta-b^2}{G-\beta+b^2}, \qquad G = (\beta^2-2a\beta+c)^{1/2}, \\ m^2 &= \frac{G+a-\beta}{2G}, \qquad m' = (1-m^2)^{1/2}. \end{aligned}$$

(4.19) may then be put in the form

$$T = \mp\frac{1}{2G^{1/2}}\int_{y(b_1^2(0))}^{y(b_1^2(T))}(1-y^2)^{-1/2}(m^2y^2+m'^2)^{-1/2}dy,$$

which is again an Elliptic Integral. The solutions are

$$b_1^2(T) = \beta - G + \frac{2G}{1+\mathrm{cn}[2G^{1/2}(\mp T+T_0);m]},$$

where

$$T_0 \;=\; \frac{1}{2G^{1/2}}\mathrm{cn}^{-1}\left[\frac{G+\beta-b_1^2(0)}{G-\beta+b_1^2(0)};m\right].$$

The above is singular at

$$T_{exp} \;=\; \frac{K}{G^{1/2}} - 2T_0,$$

where

$$K \;=\; K(m) \;=\; \int_0^1 (1-s^2)^{-1/2}(1-m^2s^2)^{-1/2}ds$$

is the quarter-period of $\mathrm{cn}[x;m]$.

We have thus proven that when the resonantly-interacting waves are such that when the first mode has wave energy opposite in sign to that of the other two, the wave amplitudes become infinite in finite time for all possible initial conditions. These interactions result from the negative energy mode losing energy to the positive energy modes so that energy conservation is obeyed. The weakly-nonlinear ansatz breaks down before the explosion takes place, and thus the $O(\epsilon^2)$ effects must be considered for times near T_{exp}. It should be noted that the existence of negative energy waves implies that energy is lost by the system in establishing the wave [6]. This is only possible if the system has some sources of free energy; in the present case, this energy is derived both from the kinetic energy of the shear, and from the free surface energy of the fluid interface.

In order to check whether or not the present system can sustain triads of the above type, one must first determine whether or not triads satisfying (4.5) exist. Since the dispersion relation (3.17) is a quartic, there are four modes for a given k, and thus 64 branches to be checked for a given triad satisfying $k_1 = k_2 + k_3$. However, only one of the neutrally-stable branches has negative wave energy, and thus the search is limited by the fact that at least one of the roots must lie on this

branch if explosive instability is to occur. Craik and Adam [9] provide an elegant graphical technique for determining the location of resonantly-interacting triads, and this technique was employed here to obtain an estimate of their values. This estimate was made more precise by finding the roots of $\omega(k_1) - \omega(k_2) - \omega(k_3) = 0$ via computer, and the values of k_2 and k_3 satisfying this condition so obtained are illustrated in Figure (4-1). The values of $\omega(k_2)$ and $\omega(k_3)$ lie on the branch labelled 1 in Figure (4-2); the energy is negative for $k \in [2.175, 2.465] \bigcup [2.55, 2.75]$, approximately, and positive elsewhere when the solutions are real. The solutions $\omega(k_1)$ lie on the branch labelled 2; all modes on this branch have positive energy, as do those on the remaining two branches. The cutoffs for k_2 and k_3 in Figure (4-1) occur because the solutions are complex for $k \in [2.465, 2.55]$, approximately.

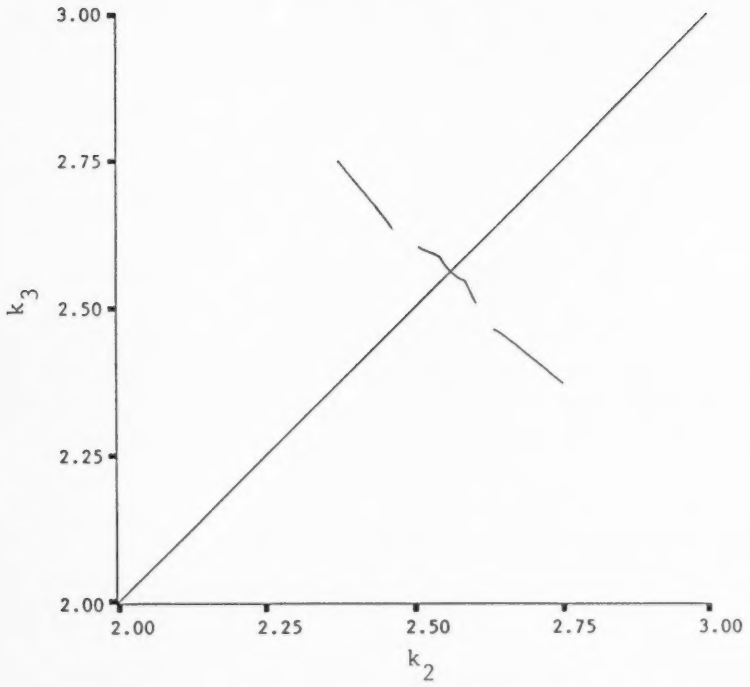

Figure (4-1): Resonant Triads ($k_1 = k_2 + k_3$): $U = 6,\ r_0 = .5,\ \rho_1 = 1.02,$ $\rho_2 = 1.015,\ \gamma = 50,\ \gamma_1 = 5,\ \gamma_2 = 0$

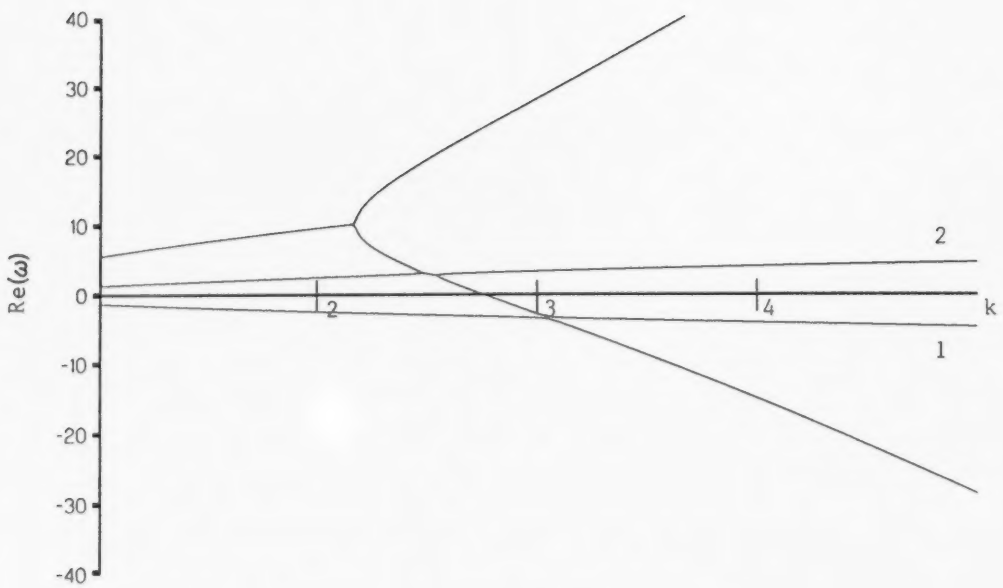

Figure (4-2): Dispersion curves $D(\omega, k) = 0$ for resonant triads: $U = 6$, $r_0 = .5$, $\rho_1 = 1.02$, $\rho_2 = 1.015$, $\gamma = 50$, $\gamma_1 = 5$, $\gamma_2 = 0$

Chapter 5

Conclusion and Discussion

The concept of negative energy waves, familiar in plasma physics, has proven useful in the study of some parallel shear flows. In the problem just examined, it was shown that both linear and nonlinear instabilities could be interpreted in terms of the coexistence of positive and negative energy waves. In the linear case, the principal effects leading to exponentially-growing solutions, and thus to the breakdown of the validity of the linear approximation, are the destabilizing effects of inertia and the surface energy released by a decrease in interfacial surface area. By contrast, the 'explosive' instabilities found to occur among resonantly-interacting triads of waves in the weakly-nonlinear treatment are due to the exchange of energy among these modes in a manner consistent with energy conservation. Both were explained, under the assumption of weak coupling between the fluid interface and tube wall, using information obtained in the linear analysis. Linear instability was shown to be the result of modes having wave energies of opposite sign coexisting on the fluid interface, while the 'explosive' nonlinear instability occurred when two negative energy waves propagated on the interface, and a positive energy wave existed on the elastic boundary. For the range of parameters examined here, the coupling between the interfaces was generally quite weak, provided that the roots of the dispersion relations for the separate interfaces were well-spaced. When they weren't, the interaction term was found to produce instability if the wave energies were of opposite sign near the point of intersection.

The linear treatment of the problem presented here was by no means complete. In the classical theory of plane parallel shear flow, general stability criteria have

been derived, giving sufficient stability and necessary instability conditions on the initial velocity profile. There is a disappointing lack of such results for cylindrical systems. As mentioned previously, there is also no analogue of Squire's Theorem for axisymmetric systems. Some special cases of the former have been proven by the author, but the boundary and interfacial conditions in the present problem prevented the derivation of such results for the general problem considered here. Further research into this area could contribute significantly to a more comprehensive understanding of fluid flow in cylindrical systems.

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Appendix I

Calculations Leading to the $O(1)$ Solutions

From (3.1b), (3.2), and (3.3),

$$p^{(0)} = -\rho(\phi_t^{(0)} + u_0\phi_x^{(0)}), \tag{I.1}$$

$$\phi_r^{(0)} = \eta_t^{(0)} + u_0\eta_x^{(0)}, \quad \text{on } r = r_0, \tag{I.2a}$$

$$\left[p^{(0)}\right] = \gamma(\eta_{xx}^{(0)} + (\eta^{(0)}/r_0^2)), \quad \text{on } r = r_0, \tag{I.2b}$$

$$\phi_r^{(0)} = a_t^{(0)}, \quad \text{on } r = 1, \tag{I.3a}$$

$$p^{(0)} = \gamma_1 a^{(0)} - \gamma_2 a_{xx}^{(0)}, \quad \text{on } r = 1, \tag{I.3b}$$

with the solutions

$$\phi^{(0)} = \begin{cases} A(T)I_0(|k|r)\exp(i\theta), & r < r_0, \\ (B(T)I_0(|k|r) + C(T)K_0(|k|r))\exp(i\theta), & r > r_0, \end{cases}$$

$$\eta^{(0)} = A_1(T)\exp(i\theta), \qquad a^{(0)} = A_2(T)\exp(i\theta).$$

(I.2a) for $r < r_0$:

$$|k|AI_1(|k|r_0) = -i(\omega - Uk)A_1,$$

$$A = -i(\omega - Uk)A_1/|k|I_1(|k|r_0). \tag{I.4}$$

(I.2b) for $r > r_0$:

$$|k|(BI_1(|k|r_0) - CK_1(|k|r_0)) = -i\omega A_1. \tag{I.5a}$$

(I.3a):

$$|k|(BI_1(|k|) - CK_1(|k|)) \;=\; -i\omega A_2. \tag{I.5b}$$

Multiply (I.5a) and (I.5b) by $I_1(|k|)$ and $I_1(|k|r_0)$, respectively, and subtract:

$$C \;=\; \frac{i\omega}{|k|P(k)}(A_2 I_1(|k|r_0) - A_1 I_1(|k|)). \tag{I.6}$$

Multiply (I.5a) and (I.5b) by $K_1(|k|)$ and $K_1(|k|r_0)$, respectively, and subtract:

$$B \;=\; \frac{i\omega}{|k|P(k)}(A_2 K_1(|k|r_0) - A_1 K_1(|k|)). \tag{I.7}$$

Appendix II

The $O(1)$ Averaged Energy Equation

Writing the full, non-dimensional momentum equations with the solutions

$$u = u_0 + \epsilon u', \quad v = \epsilon v', \quad p = p_0 + \epsilon p',$$

and the slow-time $T = \epsilon t$:

$$\begin{aligned} u_t + \epsilon u_T + u_0 u_x + \epsilon u u_x + \epsilon v u_r + \rho^{-1} p_x &= 0, \\ v_t + \epsilon v_T + u_0 v_x + \epsilon u v_x + \epsilon v v_r + \rho^{-1} p_r &= 0, \end{aligned}$$

where primes have been omitted. Multiplying the first of these by ru, the second by rv, and adding:

$$\begin{aligned} r(u^2+v^2)_t + \epsilon r(u^2+v^2)_T + r u_0 (u^2+v^2)_x + \epsilon r u (u^2+v^2)_x \qquad \text{(II.1)} \\ + \epsilon r v (u^2+v^2)_r + 2\rho^{-1}(rup)_x + 2\rho^{-1}(rvp)_r = 0, \end{aligned}$$

where

$$(rup)_x + (rvp)_r = rup_x + rvp_r,$$

by virtue of the continuity equation (2.1). We now wish to integrate over $r \in [0, 1+\epsilon a]$, which must be split into the two regions $r \in [0, r_0 + \epsilon\eta]$ and $r \in [r_0 + \epsilon\eta, 1 + \epsilon a]$. To do this, we observe that if

$$F(R) = \int_0^R f(r, x, t; T)\, dr,$$

then

$$\begin{aligned} F(r_0 + \epsilon\eta) &= F(r_0) + \epsilon\eta \left(\frac{\partial F}{\partial R}\right)_{R=r_0} + O(\epsilon^2) \\ &= F(r_0) + \epsilon\eta f(r_0, x, t; T) + O(\epsilon^2) \end{aligned}$$

A similar expression holds for the integral over $r \in [r_0 + \epsilon\eta, 1 + \epsilon a]$. Applying this to (II.1),

$$\left(\int_0^{r_0} \rho_1 + \int_{r_0}^1 \rho_2\right)\Big\{r(u^2+v^2)_t + \epsilon r(u^2+v^2)_T + r(u_0+\epsilon u)(u^2+v^2)_x$$
$$+ \epsilon r v(u^2+v^2)_r + 2\rho^{-1}(rup)_x\Big\}\,dr + 2(rvp)|_0^{r_0} + 2(rvp)|_{r_0}^1 + O(\epsilon^2) = 0.$$

Substituting the $O(1)$ solutions, one finds that the above may be written

$$\epsilon\partial_T\left\{\left(\int_0^{r_0} \rho_1 + \int_{r_0}^1 \rho_2\right) r(u^{(0)^2} + v^{(0)^2})dr\right\} + 2rv^{(0)}p^{(0)}|_0^{r_0} + 2rv^{(0)}p^{(0)}|_{r_0}^1 \qquad \text{(II.2)}$$
$$+ \{\text{terms in } \exp(\pm in\theta),\ n = 1,2,3\} + O(\epsilon^2) = 0.$$

The boundary terms are evaluated with the aid of (3.2) and (3.3):

$$\begin{aligned}(2rv^{(0)}p^{(0)})(r=1) &= 2(a_t^{(0)} + \epsilon a_T^{(0)})(\gamma_1 a^{(0)} - \gamma_2 a_{xx}^{(0)}) && \text{(II.3)}\\ &= (\partial_t + \epsilon\partial_T)(\gamma_1 a^{(0)^2} + \gamma_2 a_x^{(0)^2}\\ &\quad - 2\gamma_2(a_x^{(0)}a_t^{(0)})_x - 2\epsilon\gamma_2(a_x^{(0)}a_T^{(0)})_x,\\ 2r_0\left[v^{(0)}p^{(0)}\right] &= 2r_0(\eta_t^{(0)} + \epsilon\eta_T^{(0)})\left[p^{(0)}\right] - 2r_0 U\eta_x^{(0)}p^{(0)}(r_0^-) && \text{(II.4)}\\ &= -r_0\gamma(\partial_t + \epsilon\partial_T)(\eta_x^{(0)^2} - (\eta^{(0)^2}/r_0^2))\\ &\quad + 2r_0\gamma(\eta_x^{(0)}(\eta_t^{(0)} + \epsilon\eta_T^{(0)}))_x - 2r_0 U\eta_x^{(0)}p^{(0)}(r_0^-).\end{aligned}$$

When (II.3) and (II.4) are substituted in (II.2), and the resulting equation is averaged over $x \in [0, 2\pi/|k|]$, the terms in non-zero powers of $\exp(i\theta)$ average to zero. (II.2) reduces to

$$\epsilon\partial_T\left\{\left(\int_0^{r_0} \rho_1 + \int_{r_0}^1 \rho_2\right) r\overline{(u^{(0)^2} + v^{(0)^2})}dr + \gamma_1\overline{a^{(0)^2}} + \gamma_2\overline{a_x^{(0)^2}}\right. \qquad \text{(II.5)}$$
$$\left. + r_0\gamma\overline{(\eta_x^{(0)^2} - (\eta^{(0)^2}/r_0^2))}\right\} = 0,$$

where an overbar indicates averaging over one wavelength in x. The first term in (II.5) represents twice the rate of change of the average kinetic energy of the perturbation fields, while the remaining terms represent twice the rate at which work is done on the perturbation flow by the boundaries [15].

Appendix III
Calculations Leading to the Three-Wave Interaction Equations

From (4.4), (4.6), and (4.7):

$$\phi^{(1)} = \begin{cases} \sum_{n=1}^{3} \hat{A}^{(n)}(T) I_0(|k_n|r)\exp(i\theta_n), & r < r_0, \\ \sum_{n=1}^{3} (\hat{B}^{(n)}(T) I_0(|k_n|r) + \hat{C}^{(n)}(T) K_0(|k_n|r))\exp(i\theta_n), & r > r_0, \end{cases}$$

$$\eta^{(0)} = \sum_{n=1}^{3} A_1^{(n)}(T)\exp(i\theta_n), \qquad a^{(0)} = \sum_{n=1}^{3} A_2^{(n)}(T)\exp(i\theta_n),$$

$$\eta^{(1)} = \sum_{n=1}^{3} \hat{A}_1^{(n)}(T)\exp(i\theta_n), \qquad a^{(1)} = \sum_{n=1}^{3} \hat{A}_2^{(n)}(T)\exp(i\theta_n).$$

Observe that if

$$f = \sum_{n=1}^{3} A_n(r;T)\exp(i\theta_n), \qquad g = \sum_{n=1}^{3} B_n(r;T)\exp(i\theta_n)$$

are any two $O(1)$ solutions, then

$$fg = \frac{1}{4}\sum_{n=1}^{3}\sum_{m=1}^{3} \{A_n B_m \exp(i\theta_n + i\theta_m) + A_n B_m^* \exp(i\theta_n - i\theta_m)\} + (c.c.),$$

where $(c.c.)$ denotes complex conjugate and only the real parts of f and g are used. For notational convenience, we adopt the following:

$$f(\omega_n, k_n) = f(n), \tag{III.1}$$

$$\mathcal{D}_1(n) = \frac{k_n r_0}{\rho_2 \omega_n}\left\{\frac{\rho_1(\omega_n - Uk_n)^2 I_0(|k_n|r_0)}{|k_n| I_1(|k_n|r_0)} + \gamma(r_0^{-2} - k_n^2)\right\} \tag{III.2a}$$

$$= \frac{k_n}{\rho_2\omega_n}\left\{D_1(n) + \frac{r_0\rho_2\omega_n^2 F(k_n)}{|k_n|P(k_n)}\right\}, \tag{III.2b}$$

$$\mathcal{D}_2(n) = \frac{-k_n}{\rho_2\omega_n}(\gamma_1 + k_n^2\gamma_2) \tag{III.3a}$$

$$= \frac{k_n}{\rho_2\omega_n}\left\{D_2(n) + \frac{\rho_2\omega_n^2 G(k_n)}{|k_n|P(k_n)}\right\}, \tag{III.3b}$$

and

$$\overline{f}(k_n,\omega_n,k_m,\omega_m,A_1^{(n)},A_1^{(m)}) \;=\; f(k_n,\omega_n,-k_m,-\omega_m,A_1^{(n)},A_1^{(m)^*}). \quad \text{(III.4)}$$

The notation $f(r_0^-)$ is used to denote a field interior to the fluid interface evaluated at $r = r_0$, and similarly for $f(r_0^+)$. Finally, both * and $(c.c.)$ denote complex conjugate.

Referring to (4.2) and (4.3) in Chapter 4, we obtain the following:

(4.2a) for $r < r_0$:

$$\begin{aligned}
\phi_r^{(1)}(r_0^-) &= \eta_t^{(1)} + \eta_T^{(0)} + U\eta_x^{(1)} + R(\theta), \\
R(\theta) &= -\eta^{(0)}\phi_{rr}^{(0)}(r_0^-) + \eta_x^{(0)}\phi_x^{(0)}(r_0^-) \\
&= \sum_{n=1}^{3}\sum_{m=1}^{3} R_{nm}\exp(i\theta_n + i\theta_m) + \overline{R}_{nm}\exp(i\theta_n - i\theta_m) + (c.c.),
\end{aligned}$$

$$R_{nm} = \frac{i(\omega_n - Uk_n)\left\{r_0k_n(k_n+k_m)I_0(|k_n|r_0) - |k_n|I_1(|k_n|r_0)\right\}}{4r_0|k_n|I_1(|k_n|r_0)}A_1^{(n)}A_1^{(m)}, \quad \text{(III.5a)}$$

$$\Rightarrow \hat{A}^{(n)} \;=\; -\frac{i(\omega_n - Uk_n)\hat{A}_1^{(n)} + A_{1_T}^{(n)} + R(\theta_n)}{|k_n|I_1(|k_n|r_0)}, \quad \text{(III.5b)}$$

where $R(\theta_n)$ denotes those terms in $R(\theta)$ having phase equal to θ_n. This notation is used throughout this appendix.

(4.2a) for $r > r_0$:

$$\begin{aligned}
\phi_r^{(1)}(r_0^+) &= \eta_t^{(1)} + \eta_T^{(0)} + S(\theta), \\
S(\theta) &= -\eta^{(0)}\phi_{rr}^{(0)}(r_0^+) + \eta_x^{(0)}\phi_x^{(0)}(r_0^+) \\
&= \sum_{n=1}^{3}\sum_{m=1}^{3} S_{nm}\exp(i\theta_n + i\theta_m) + \overline{S}_{nm}\exp(i\theta_n - i\theta_m) + (c.c.),
\end{aligned}$$

$$S_{nm} \;=\; \frac{i((k_n+k_m)\mathcal{D}_1(n) - \omega_n)}{4r_0}A_1^{(n)}A_1^{(m)}, \quad \text{(III.6a)}$$

$$\Rightarrow |k_n|(\hat{B}^{(n)} I_1(|k_n|r_0) - \hat{C}^{(n)} K_1(|k_n|r_0)) = -i\omega_n \hat{A}_1^{(n)} + A_{1_T}^{(n)} + S(\theta_n). \quad \text{(III.6b)}$$

(3.3a):

$$\begin{aligned} \phi_r^{(1)}(r=1) &= a_t^{(1)} + a_T^{(0)} + T(\theta), \\ T(\theta) &= -a^{(0)}\phi_{rr}^{(0)}(r=1) + a_x^{(0)}\phi_x^{(0)}(r=1) \\ &= \sum_{n=1}^{3}\sum_{m=1}^{3} T_{nm}\exp(i\theta_n + i\theta_m) + \overline{T}_{nm}\exp(i\theta_n - i\theta_m) + (c.c.), \end{aligned}$$

$$T_{nm} = \frac{-iD_1(n)D_1(m)((k_n + k_m)\mathcal{D}_2(n) + \omega_n)}{4\Lambda(n)\Lambda(m)} A_1^{(n)} A_1^{(m)}, \quad \text{(III.7a)}$$

$$\Rightarrow |k_n|(\hat{B}^{(n)} I_1(|k_n|) - \hat{C}^{(n)} K_1(|k_n|)) = -i\omega_n \hat{A}_2^{(n)} + A_{2_T}^{(n)} + T(\theta_n). \quad \text{(III.7b)}$$

Equations (III.6b) and (III.7b) may be combined to write

$$\hat{B}^{(n)} = \frac{K_1(|k_n|)\sigma_n - K_1(|k_n|r_0)\tau_n}{|k_n|P(k_n)}, \quad \text{(III.8)}$$

$$\hat{C}^{(n)} = \frac{I_1(|k_n|)\sigma_n - I_1(|k_n|r_0)\tau_n}{|k_n|P(k_n)}, \quad \text{(III.9)}$$

where

$$\begin{aligned} \sigma_n &= -i\omega_n \hat{A}_1^{(n)} + A_{1_T}^{(n)} + S(\theta_n), \\ \tau_n &= -i\omega_n \hat{A}_2^{(n)} + A_{2_T}^{(n)} + T(\theta_n). \end{aligned}$$

(4.2b):

$$-\left[\rho(\phi_t^{(1)} + u_0\phi_x^{(1)} + \phi_T^{(0)})\right] = \gamma(\eta_{xx}^{(1)} + \eta^{(1)}/r_0^2) + Q(\theta), \quad \text{(III.10)}$$

$$\begin{aligned} Q(\theta) &= \gamma(\eta_x^{(0)^2}/2r_0 - \eta^{(0)^2}/r_0^3) + \left[\rho\eta^{(0)}(\phi_{tr}^{(0)} + u_0\phi_{xr}^{(0)}) + \rho(\nabla\phi^{(0)})^2/2\right] \\ &= \sum_{n=1}^{3}\sum_{m=1}^{3} Q_{nm}\exp(i\theta_n + i\theta_m) + \overline{Q}_{nm}\exp(i\theta_n - i\theta_m) + (c.c.), \end{aligned}$$

$$
\begin{aligned}
Q_{nm} &= \frac{1}{4}\left\{-\rho_2\omega_n^2+\rho_1(\omega_n-Uk_n)^2+\frac{\rho_2}{2}\left(\frac{\mathcal{D}_1(n)\mathcal{D}_1(m)}{r_0^2}-\omega_n\omega_m\right)\right. \\
&\quad -\frac{\rho_1}{2}(\omega_n-Uk_n)(\omega_m-Uk_m)\left(\frac{k_nk_mI_0(|k_n|r_0)I_0(|k_m|r_0)}{|k_nk_m|I_1(|k_n|r_0)I_1(|k_m|r_0)}-1\right) \\
&\quad \left.-\gamma\left(\frac{k_nk_m}{2r_0}+\frac{1}{r_0^3}\right)\right\}A_1^{(n)}A_1^{(m)}.
\end{aligned}
$$

(4.3b):

$$
-\rho_2\phi_t^{(1)}(r=1) = \gamma_1a^{(1)}-\gamma_2a_{xx}^{(1)}+\rho_2\phi_T^{(0)}+P(\theta), \tag{III.11}
$$

$$
\begin{aligned}
P(\theta) &= \gamma_3a^{(0)^2}+\gamma_4a^{(0)}a_{xx}^{(0)}+\gamma_5a_x^{(0)^2} \\
&\quad +\rho_2a^{(0)}\phi_{rt}^{(0)}(r=1)+\frac{\rho_2}{2}(\nabla\phi^{(0)})^2(r=1) \\
&= \sum_{n=1}^{3}\sum_{m=1}^{3}P_{nm}\exp(i\theta_n+i\theta_m)+\overline{P}_{nm}\exp(i\theta_n-i\theta_m)+(c.c.),
\end{aligned}
$$

$$
P_{nm}=\frac{D_1(n)D_1(m)}{4\Lambda(n)\Lambda(m)}\left\{\frac{\rho_2}{2}(\mathcal{D}_2(n)\mathcal{D}_2(m)-\omega_n\omega_m-2\omega_n^2)+\gamma_3-\gamma_4k_n^2+\gamma_5k_nk_m\right\}A_1^{(n)}A_1^{(m)}.
$$

Derivation of the Three-Wave Interaction Equations

The three-wave interaction equations follow directly from (III.10) and (III.11) above. From (III.5b), (III.8), and (III.9), substitute for $\hat{A}^{(n)}$, $\hat{B}^{(n)}$, and $\hat{C}^{(n)}$ in (III.10) to obtain

$$
D_1(n)\hat{A}_1^{(n)}+\Lambda(n)\hat{A}_2^{(n)} = \Phi(n)A_{1_T}^{(n)}+\Gamma(n), \tag{III.12}
$$

where (3.10) and (3.14) have been used to eliminate $A_2^{(n)}$ in favour of $A_1^{(n)}$, and

$$\Phi(n) = ir_0\left\{\frac{\rho_2\omega_n F(k_n)}{|k_n|P(k_n)} + \frac{\rho_2\mathcal{D}_1(n)}{k_n r_0} + \frac{\rho_2\omega_n D_1(n)}{k_n^2 r_0 P(k_n)\Lambda(n)} - \frac{2\rho_1(\omega_n - Uk_n)I_0(|k_n|r_0)}{|k_n|I_1(|k_n|r_0)}\right\}, \tag{III.13a}$$

$$\Gamma(n) = ir_0\left\{\frac{\rho_2\omega_n F(k_n)S(\theta_n)}{|k_n|P(k_n)} - \frac{\rho_2\omega_n T(\theta_n)}{k_n^2 r_0 P(k_n)} - \frac{\rho_1(\omega_n - Uk_n)I_0(|k_n|r_0)R(\theta_n)}{|k_n|I_1(|k_n|r_0)} + iQ(\theta_n)\right\}. \tag{III.13b}$$

Doing the same in (III.11):

$$D_2(n)\hat{A}_2^{(n)} + \Lambda(n)\hat{A}_1^{(n)} = \chi(n)A_{1_T}^{(n)} + \Omega(n), \tag{III.14}$$

where

$$\chi(n) = -i\rho_2\left\{\frac{\omega_n}{k_n^2 P(k_n)} + \frac{\mathcal{D}_2(n)D_1(n)}{k_n\Lambda(n)} + \frac{\omega_n G(k_n)D_1(n)}{|k_n|P(k_n)\Lambda(n)}\right\}, \tag{III.15a}$$

$$\Omega(n) = -\frac{i\rho_2\omega_n}{|k_n|P(k_n)}\left\{\frac{S(\theta_n)}{|k_n|} - T(\theta_n)G(k_n)\right\} + P(\theta_n). \tag{III.15b}$$

Combining (III.12) and (III.14), and using the fact that $D_1(n)D_2(n) - \Lambda^2(n) = 0$, we have

$$(\Phi(n)\Lambda(n) - D_1(n)\chi(n))A_{1_T}^{(n)} = D_1(n)\Omega(n) - \Lambda(n)\Gamma(n). \tag{III.16}$$

But by (III.2) and (III.3),

$$\begin{aligned}\Phi(n)\Lambda(n) - D_1(n)\chi(n) &= \frac{i\Lambda(n)}{\omega_n}\left\{\frac{2r_0\rho_2\omega_n^2 F(k_n)}{|k_n|P(k_n)} + \frac{2\rho_2\omega_n^2 G(k_n)D_1(n)}{|k_n|D_2(n)P(k_n)} - \frac{2r_0\rho_1\omega_n(\omega_n - Uk_n)I_0(|k_n|r_0)}{|k_n|I_1(|k_n|r_0)} + 4D_1(n)\right\} \\ &= -i\Lambda(n)\frac{\partial D(n)}{\partial\omega_n},\end{aligned} \tag{III.17}$$

by virtue of (3.11), (3.15), and (3.21). The expression for the right-hand side of (III.16) is much more difficult to obtain. We shall select the case $n = 1$; the procedures for $n = 2$ and $n = 3$ are identical.

First, we observe from the resonance condition $\theta_1 = \theta_2 + \theta_3$ that $P(\theta_1) = P_{23} + P_{32}$, with similar expressions for the other nonlinear terms. Substituting for these quantities in (III.13b) and (III.15b) and using (III.2) and (III.3) in the result, one finds

$$
\begin{aligned}
D_1(1)\Omega_1(1) - \Lambda(1)\Gamma(1) = & \left\{ \frac{\rho_2 \Lambda(1)\mathcal{D}_1(1)}{4r_0 k_1}(k_1(\mathcal{D}_1(2) + \mathcal{D}_1(3)) - \omega_1) \right. \qquad \text{(III.18)} \\
& + \frac{\Lambda(1)\Theta}{4}\left\{ \rho_2 \mathcal{D}_2(2)\mathcal{D}_2(3) + 2\gamma_3 - \gamma_4(k_2^2 + k_3^2) + 2\gamma_5 k_2 k_3 - \rho_2(\omega_2^2 + \omega_3^2 + \omega_2\omega_3) \right\} \\
& + \frac{\rho_2 \Lambda(1)\Theta\mathcal{D}_2(1)}{4k_1}(k_1(\mathcal{D}_2(2) + \mathcal{D}_2(3)) + \omega_1) \\
& - \frac{r_0\rho_1\Lambda(1)(\omega_1 - Uk_1)I_0(|k_1|r_0)}{4|k_1|I_1(|k_1|r_0)}\left\{ \frac{k_1k_2(\omega_2 - Uk_2)I_0(|k_2|r_0)}{|k_2|I_1(|k_2|r_0)} \right. \\
& \quad \left. + \frac{k_1k_3(\omega_3 - Uk_3)I_0(|k_3|r_0)}{|k_3|I_1(|k_3|r_0)} - \frac{(\omega_1 - Uk_1)}{r_0} \right\} \\
& + \frac{r_0\Lambda(1)}{4}\left\{ -2\gamma\left(\frac{k_2k_3}{2r_0} + \frac{1}{r_0^3}\right) + \frac{\rho_2\mathcal{D}_1(2)\mathcal{D}_1(3)}{r_0^2} \right. \\
& \quad - \rho_2(\omega_2^2 + \omega_3^2 + \omega_2\omega_3) + \rho_1(\omega_2 - Uk_2)^2 + \rho_1(\omega_3 - Uk_3)^2 \\
& \quad \left. \left. - \rho_1(\omega_2 - Uk_2)(\omega_3 - Uk_3)\left(\frac{k_2k_3I_0(|k_2|r_0)I_0(|k_3|r_0)}{|k_2k_3|I_1(|k_2|r_0)I_1(|k_3|r_0)} - 1 \right) \right\} \right\} A_1^{(2)}A_1^{(3)},
\end{aligned}
$$

where we have introduced

$$
\Theta \;=\; \frac{D_1(1)D_1(2)D_1(3)}{\Lambda(1)\Lambda(2)\Lambda(3)},
$$

and the resonance conditions (4.5) have been used throughout. Before writing the above, all terms in $P(\theta_1)$, $Q(\theta_1)$, $R(\theta_1)$, $S(\theta_1)$, and $T(\theta_1)$ were gathered together; there is thus one term for each in (III.18). Upon defining

$$
\mathcal{F}(i,j) \;=\; -\frac{r_0\rho_1 k_i k_j(\omega_i - Uk_i)(\omega_j - Uk_j)I_0(|k_i|r_0)I_0(|k_j|r_0)}{|k_ik_j|I_1(|k_i|r_0)I_1(|k_j|r_0)},
$$

the right-hand side of (III.18) can be written

$$
\begin{aligned}
& \frac{\Lambda(1)}{4}\left\{ \mathcal{V} + \frac{\rho_2\omega_1\mathcal{D}_2(1)}{k_1} - \gamma(r_0^{-2} - k_1^2) \right. \qquad \text{(III.19)} \\
& \quad \left. + \Theta\left(2\gamma_3 - \gamma_4(k_2^2 + k_3^2) - 2\gamma_5 k_2 k_3 - \rho_2(\omega_2^2 + \omega_3^2 + \omega_2\omega_3)\right) \right.
\end{aligned}
$$

$$+ \rho_1 r_0\{(\omega_2 - Uk_2)^2 + (\omega_3 - Uk_3)^2 + (\omega_2 - Uk_2)(\omega_3 - Uk_3)\}$$
$$- 2\gamma r_0 \left(\frac{k_2 k_3}{2r_0} + \frac{1}{r_0^3}\right) - \rho_2 r_0(\omega_2^2 + \omega_3^2 + \omega_2\omega_3)\Big\} A_1^{(2)} A_1^{(3)}.$$

In the above, $\mathcal{V}$ is the term symmetric in $(1,2,3)$ given by

$$\begin{aligned}\mathcal{V} &= \frac{\rho_2}{r_0}\Big\{\mathcal{D}_1(1)\mathcal{D}_1(2) + \mathcal{D}_1(1)\mathcal{D}_1(3) + \mathcal{D}_1(2)\mathcal{D}_1(3)\Big\} \\ &\quad + \rho_2\Theta\{\mathcal{D}_2(1)\mathcal{D}_2(2) + \mathcal{D}_2(1)\mathcal{D}_2(3) + \mathcal{D}_2(2)\mathcal{D}_2(3)\} \\ &\quad + \mathcal{F}(1,2) + \mathcal{F}(1,3) + \mathcal{F}(2,3).\end{aligned}$$

When (III.19) is derived for either of the cases $n = 2$ or $n = 3$, it is found that there are terms of the form $\omega_1^2 + \omega_3^2 - \omega_1\omega_3$; use of the resonance conditions shows that this is in fact equal to $\omega_2^2 + \omega_3^2 + \omega_2\omega_3$. Hence, such expressions may also be considered symmetric. It follows from the definitions of γ_3, γ_4, and γ_5 given in Chapter 4, and from (III.3a) that

$$\begin{aligned}2\gamma_3 - \gamma_4(k_2^2 + k_3^2) + 2\gamma_5 k_2 k_3 &= \frac{\rho_2\omega_2\mathcal{D}_2(2)}{k_2} + \frac{\rho_2\omega_3\mathcal{D}_2(3)}{k_3} \\ &\quad + W_{12}^0(k_2^2 + k_3^2 + k_2k_3) + \frac{W_{111}^0}{(1+e)},\end{aligned}$$

and thus (III.19) becomes

$$D_1(1)\Omega(1) - \Lambda(1)\Gamma(1) = \frac{\Lambda(1)}{4}\Big(\mathcal{V}' + \gamma(k_1^2 - k_2k_3)\Big) A_1^{(2)} A_1^{(3)}, \quad \text{(III.20)}$$

$$\begin{aligned}\mathcal{V}' &= \mathcal{V} - \frac{3\gamma}{r_0^2} + \rho_2\Theta\left\{\frac{\omega_1\mathcal{D}_2(1)}{k_1} + \frac{\omega_2\mathcal{D}_2(2)}{k_2} + \frac{\omega_3\mathcal{D}_2(3)}{k_3}\right\} \\ &\quad - \rho_2(\Theta + r_0)(\omega_2^2 + \omega_3^2 + \omega_2\omega_3) + \Theta\left\{W_{12}^0(k_2^2 + k_3^2 + k_2k_3) + \frac{W_{111}^0}{(1+e)}\right\} \\ &\quad + r_0\rho_1\{(\omega_2 - Uk_2)^2 + (\omega_3 - Uk_3)^2 + (\omega_2 - Uk_2)(\omega_3 - Uk_3)\}.\end{aligned}$$

The remaining term apart from $\mathcal{V}'$ in the above takes the form $k_2^2 + k_1k_3$ and $k_3^2 + k_1k_2$ for $n = 2$ and $n = 3$, respectively. Again, the resonance conditions imply

the equality of both of these expressions to $k_1^2 - k_2 k_3$. Thus, (III.20) may be written as $\Lambda(1)\lambda A_1^{(2)} A_1^{(3)}$, with λ given by the symmetric expression

$$\lambda \;=\; \frac{1}{4}\Big(\mathcal{V}' + \gamma(k_1^2 - k_2 k_3) \Big) . \qquad \text{(III.21)}$$

Combining this with (III.17), we obtain the evolution equation for $A_1^{(1)}$:

$$-i \frac{\partial D(1)}{\partial \omega_1} A_{1_T}^{(1)} \;=\; \lambda A_1^{(2)} A_1^{(3)} .$$